Answers to Exercise 5

I.

Proof . $\{m^*$ is translation invariant, it means that for any $a \in \mathbb{R}, E \subset \mathbb{R}, a+E = \{a+x|\, x \in E\}$, then $m^*(a+E) = m^*(E)\}$.

$E \subset \mathbb{R}, m^*(E) = \inf\left\{ \sum_n l(I_n)|\, I_n \text{ is open intervals such that } E \subset \bigcup_n I_n \right\}$. Let's denote $I'_n = a + I_n = \{a+x|\, x \in I_n\}$, for any intervals $\{I_n\}_n$, s.t. $E \subset \bigcup_n I_n$. Then it is obvious that $a + E \subset \bigcup_n I'_n$. Further, for any n, $l(I'_n) = \sup\{|x'-y'|, x', y' \in I'_n\} = \sup\{|(a+x)-(a+y)|, x, y \in I_n\} = \sup\{|x-y|, x, y \in I_n\} = l(I_n)$, so $m^*(a+E) = \inf\left\{ \sum_n l(I'_n)|\, E \subset \bigcup_n I'_n \right\} = \inf\left\{ \sum_n l(I_n)|\, E \subset \bigcup_n I_n \right\} = m^*(E)$.

II.

Proof . $\{f_n\} \longrightarrow f$ means that $\sup\{|f_n(x) - f(x)| \mid x \in M\} \longrightarrow 0$. Put in another way, for any $\varepsilon > 0$, there exists a $N_1 \in \mathbb{N}$, when $n \geq N_1$, $|f_n(x) - f(x)| < \varepsilon$, for all $x \in M$. The same to g_n, for convenience, we may choose the same ε as above, then there also exists a $N_2 \in \mathbb{N}$, when $n \geq N_2$, $|g_n(x) - g(x)| < \varepsilon$, for all $x \in M$. Since the space $C(M)$ is complete, $f \in C(M)$, and $g \in C(M)$, then $f + g \in C(M)$, we will show that $\{f_n + g_n\} \longrightarrow f + g$ in $C(M)$. For any $\varepsilon > 0$, denote $N = \max\{N_1, N_2\}$, when $n \geq N$,

$$\begin{aligned} \big|\big(f_n(x) + g_n(x)\big) - \big(f(x) + g(x)\big)\big| &= \big|\big(f_n(x) - f(x)\big) + \big(g_n(x) - g(x)\big)\big| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < 2\varepsilon, \end{aligned}$$

which equals to say that $\sup\{|\big(f_n(x) + g_n(x)\big) - \big(f(x) + g(x)\big)| \mid x \in M\} \longrightarrow 0$. So $\{f_n + g_n\} \longrightarrow f + g$ uniformly.

Next we will show that $\{f_n\}$ is **uniformly** bounded on M, when n large enough. Let $\varepsilon = 1$, then there exists $N' \in \mathbb{N}$, when $n \geq N'$, $\sup_{x\in M} |f_n(x) - f(x)| < 1$, so

$$\begin{aligned} \sup_{x\in M} |f(x)| &= \sup_{x\in M} \{|f(x) - f_{N'}(x) + f_{N'}(x)|\} \\ &\leq \sup_{x\in M} \{|f_{N'}(x) - f(x)| + |f_{N'}(x)|\} \\ &\leq \sup_{x\in M} \{|f_{N'}(x) - f(x)|\} + \sup_{x\in M} \{|f_{N'}(x)|\} \\ &\leq 1 + K = D_1 \end{aligned}$$

$\big($Note: it is easy to show that $\sup(A+B) \leq \sup A + \sup B\big)$, for some finite constant $K = \sup_{x\in M} |f_{N'}(x)|, x \in M$. It is the same to g, that is there also exists D_2, s.t. $|g(x)| \leq D_2$ for all $x \in M$.

Now for $n \geq N'$,

$$\begin{aligned} \sup_{x\in M} |f_n(x)| &= \sup_{x\in M} |f_n(x) - f(x) + f(x)| \\ &\leq \sup_{x\in M} |f_n(x) - f(x)| + \sup_{x\in M} |f(x)| \\ &\leq 1 + 1 + K = 2 + K = C_1\,. \end{aligned}$$

It is the same to $\{g_n\}$. There also exists some constant, and N'', s.t. when $n \geq N''$, $\sup_{x\in M} |g_n(x)| \leq C_2$.

Since $\{f_n\}$, and $\{g_n\}$ are convergent uniformly on M, for any $\varepsilon > 0$, there exists n_1, n_2, when $n, m \geq n^* = \max\{n_1, n_2\}$, $|f_n(x) - f_m(x)| < \varepsilon$ and $|g_n(x) - g_m(x)| < \varepsilon$, for all $x \in M$.

Then for $N^* = \max\{N', N'', n_1, n_2\}$, when $n, m \geq N^*$, for all $x \in M$,

$$\begin{aligned} |f_n g_n - f_m g_m| &= |(f_n g_n - f_n g_m) + (f_n g_m - f_m g_m)| \\ &\leq |f_n(g_n - g_m)| + |g_m(f_n - f_m)| = |f_n||g_n - g_m| + |g_m||f_n - f_m| \\ &\leq C_1 \cdot \varepsilon + C_2 \cdot \varepsilon\,. \end{aligned}$$

thus $\{f_n g_n\}$ converges uniformly on M.

Actually, since $C(M)$ is complete, there must exist a function $H \in C(M)$, s.t. $\{f_n g_n\} \longrightarrow H$ uniformly, we show that $H = fg$. Because for all $x \in M$, and for all $n \geq N^{\#} = \max\{N_1, N_2, N'\}$,

$$\begin{aligned} |f_n g_n - fg| &= |f_n g_n - f_n g + f_n g - fg| \leq |f_n(g_n - g)| + |(f_n - f)g| \\ &= |f_n||g_n - g| + |f_n - f||g| \\ &\leq C_1 \cdot \varepsilon + D_2 \cdot \varepsilon\,, \end{aligned}$$

which means $\{f_n g_n\} \longrightarrow fg$ uniformly, and from the uniqueness of limit, we know $H = fg$.

III.

Proof . Apply the definition of uniform convergence directly.

$$\sup\left\{|f_n(x) - 0| \mid x \in [0, \infty[\ \right\} \leq \sup\left\{\frac{1}{n}\middle| x \in [0, \infty[\right\} = \frac{1}{n} \longrightarrow 0.$$

IV.

Proof . From theorem 2.3,

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(A).$$

On the other side, $A \subset A \cup B$, from the definition of outer measure,

$$m^*(A) \leq m^*(A \cup B).$$

So $m^*(A \cup B) = m^*(A)$, if $m^*(B) = 0$.

V.

Proof . We know that

$$E_1 \cup E_2 = (E_1 \backslash E_2) \bigcup (E_1 \cap E_2) \bigcup (E_2 \backslash E_1),$$

since the three parts on the left are disjoint, we have

$$m(E_1 \cup E) = m(E_1 \backslash E_2) + m(E_1 \cap E_2) + m(E_2 \backslash E_1)\,.$$

Then

$$\begin{aligned} m(E_1 \cup E) + m(E_1 \cap E_2) &= \Big[m(E_1 \backslash E_2) + m(E_1 \cap E_2)\Big] + \Big[m(E_2 \backslash E_1) + m(E_1 \cap E_2)\Big] \\ &= m(E_1) + m(E_2). \end{aligned}$$

VI.

Proof . First we proof

$$m^*\Big(A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\Big)=\sum_{i=1}^{n}m^*(A\cap E_i)\,.$$

We prove the lemma by induction on n. It is clearly true for $n=1$, and we assume it is true if we have $n-1$ sets E_i. Since the E_i are disjoint sets, we have

$$A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\cap E_n=A\cap E_n$$

and

$$A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\cap E_n^-=A\cap\Big[\bigcup_{i=1}^{n-1}E_i\Big].$$

Hence the measurability of E_n implies

$$\begin{aligned} m^*\Big(A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\Big) &= m^*\Big(A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\cap E_n\Big)+m^*\Big(A\cap\Big[\bigcup_{i=1}^{n}E_i\Big]\cap E_n^-\Big)\\ &= m^*(A\cap E_n)+m^*\Big(A\cap\Big[\bigcup_{i=1}^{n-1}E_i\Big]\Big)\\ &= m^*(A\cap E_n)+\sum_{i=1}^{n-1}m^*(A\cap E_i)\\ &= \sum_{i=1}^{n}m^*(A\cap E_i)\end{aligned}$$

by our assumption of the lemma for $n-1$ sets.

Next we consider the infinite union.

Since $\bigcup_{i=1}^{\infty}E_i\supset\bigcup_{i=1}^{n}E_i$, for any set A, $A\cap\bigcup_{i=1}^{\infty}E_i\supset A\cap\bigcup_{i=1}^{n}E_i$, then

$$m^*(A\cap\bigcup_{i=1}^{\infty}E_i)\ge m^*(A\cap\bigcup_{i=1}^{n}E_i)=\sum_{i=1}^{n}m^*(A\cap E_i)\,,$$

which holds for any n, so

$$m^*(A\cap\bigcup_{i=1}^{\infty}E_i)\ge\sum_{i=1}^{\infty}m^*(A\cap E_i)\,.$$

On the other side, since

$$A\cap\bigcup_{i=1}^{n}E_i=\bigcup_{i=1}^{n}A\cap E_i\,,$$

we know for all n,

$$m^*\Big(A \cap \bigcup_{i=1}^{n} E_i\Big) = m^*\Big(\bigcup_{i=1}^{n} A \cap E_i\Big) = \sum_{i=1}^{n} m^*(A \cap E_i) \le \sum_{i=1}^{\infty} m^*(A \cap E_i)\,,$$

for $\{E_i\}$ are disjoint, $\{A \cap E_i\}$ are disjoint. Then

$$m^*\Big(A \cap \bigcup_{i=1}^{\infty} E_i\Big) \le \sum_{i=1}^{\infty} m^*(A \cap E_i)\,.$$

Thus

$$m^*\Big(A \cap \Big[\bigcup_{i=1}^{\infty} E_i\Big]\Big) = \sum_{i=1}^{\infty} m^*(A \cap E_i)\,.$$