Answers to Exercise 6

I.

Proof. Suppose $f : E \longrightarrow \mathbb{R}$. Since f is measurable function, according to the definition, for any $r \in \mathbb{R}$, $\{x \in E|\, f(x) \geq r\}$ and $\{x \in E|\, f(x) \leq r\}$, are all measurable sets. Thus, from theorem 2.12, we know that $f^{-1}\{r\}$ is measurable, for $f^{-1}(\{r\}) = \{x \in E|\, f(x) \geq r\} \cap \{x \in E|\, f(x) \leq r\}$.

Let $I \subset \mathbb{R}$ be any interval, we may as well suppose $I = (a, b), a, b \in R$, then $f^{-1}(\{a < f(x) < b\}) = \{x \in E|\, f(x) > a\} \cap \{x \in E|\, f(x) < b\}$. Again from the fact that f is measurable, we know that both $\{x \in E|\, f(x) > a\}$ and $\{x \in E|\, f(x) < b\}$ are measurable sets, so does $\{x \in E|\, f(x) > a\} \cap \{x \in E|\, f(x) < b\}$.

II.

Proof. We may as well suppose that $f \geq g \geq 0$. Let ϕ be a simple function, s.t. $0 \leq \phi \leq g$, then $0 \leq \phi \leq f$, and $\int \phi dm \leq \int f dm$ (from the definition 2.24). Since ϕ is any simple function, which satisfies $\phi \leq g$,

$$\int g dm = \sup\{\int \phi dm|\, \phi \,\text{simple and}\, \phi \leq g\} \leq \int f dm,$$

holds for any function f and g. Because $f \geq g$, so $f^+ \geq g^+$ and $f^- \leq g^-$. Further,

$$\int f^+ dm - \int f^- dm$$

is defined, since $\int f dm$ exists. And from the fact that $\int f dm < \infty$, we know $\int f^+ dm < \infty$, thus $\int g^+ dm < \int f^+ dm < \infty$, Therefore $\int g dm$ is exist, and we can get

$$\int g dm = \int g^+ dm - \int g^- dm \leq \int f^+ dm - \int f^- dm = \int f dm.$$

III.

Proof. Since f and g are measurable functions, according to the theorem 2.17, $f - g$ is measurable. Then

$$\{x \in E|\, f(x) < g(x)\} = \{x \in E|\, (f - g)(x) < 0\}$$

is measurable. Similarly,

$$\{x \in E|\, f(x) \leq g(x)\} = \{x \in E|\, (f - g)(x) \leq 0\}$$

is measurable. And from question 1, we can easily get

$$\{x \in E|\, f(x) = g(x)\} = \{x \in E|\, (f - g)(x) = 0\} = f^{-1}(\{0\})$$

is measurable.

IV.

Proof. Let $G = \max\{f_1, \cdots, f_n\}$, and $H = \min\{f_1, \cdots, f_n\}$. For any $r \in \mathbb{R}$,

$$A_1 = \{x \in E|\, G(x) > r\} = \bigcup_{i=1}^{n} \{x \in E|\, f_i(x) > r\},$$

and

$$A_2 = \{x \in E |\, H(x) > r\} = \bigcap_{i=1}^{n} \{x \in E |\, f_i(x) > r\}\,.$$

Since each $f_i, i = 1, \cdots, n$ is measurable function, $\{x \in E |\, f_i(x) > r\}$ is measurable, for any $r \in \mathbb{R}$. By theorem 2.12, we know that A_1 and A_2 are measurable, thus G and H are measurable functions.

V.

Proof. **AT**. If not, for the function $f : E \longrightarrow R$, we may as well suppose $f(x) > 0$ holds for $\forall x \in A \subset E$, and $m(A) > 0$.

$$A = \bigcup_{i=1}^{\infty} \left\{x \in A \middle| \frac{1}{i+1} \leq f(x) < \frac{1}{i}\right\} \bigcup \{x \in A |\, 1 \leq f(x)\} = \bigcup_{i=1}^{\infty} A_i \bigcup A_\infty\,.$$

Since for all sets $\left\{x \in A \middle| \frac{1}{i+1} \leq f(x) < \frac{1}{i}\right\}, i \in \mathbb{N}$ and the set $\{x \in A |\, 1 \leq f(x)\}$ are disjoint, we get

$$m(A) = \sum_{i=1}^{\infty} m(A_i) + m(A_\infty) > 0,$$

Thus $m(A_i) > 0$ must hold for some $i \in \mathbb{N}$ or $i = \infty$. Then $\int\limits_E f dm \geq \int\limits_A f dm \geq \int\limits_{A_i} f dm \geq \frac{1}{i+1} m(A_i) > 0$, a contradiction.