Answers to Exercise 8

I.

Proof. $f \in L^\infty$,

$$b_1 = ess\sup |f(x)| = \inf\{b|\,|f(x)| \le b, a.e.\} < \infty.$$

It means for any $\varepsilon > 0$, there exists a set $E_1 \in \mathcal{M}$, for any $x \in E_1$, $|f(x)| \le b_1 + \varepsilon$, and $m(\mathbb{R}\backslash E_1) = 0$.

Similarly, for $g \in L^\infty$,

$$b_2 = ess\sup |g(x)| = \inf\{b|\,|g(x)| \le b, a.e.\} < \infty.$$

For the same ε, there also exists a set $E_2 \in \mathcal{M}$, s.t. for any $x \in E_2$, $|g(x)| \le b_2 + \varepsilon$, and $m(\mathbb{R}\backslash E_2) = 0$.

Then $m(\mathbb{R}\backslash E_1 \cap E_2) = m(\mathbb{R}\backslash E_1 \cup \mathbb{R}\backslash E_2) \le m(\mathbb{R}\backslash E_1) + m(\mathbb{R}\backslash E_2) = 0$, for $x \in E_1 \cap E_2$,

$$|f(x)g(x)| = |f(x)||g(x)| \le (b_1 + \varepsilon)(b_2 + \varepsilon) = b_1 b_2 + (b_1 + b_1 + \varepsilon)\varepsilon$$

holds for any $\varepsilon > 0$, so

$$ess\sup |f(x)g(x)| = \inf\{b|\,|f(x)g(x)| \le b, a.e.\} \le b_1 b_2 < \infty\,,$$

which means $fg \in L^\infty$.

II.

Proof. **AT.** $\{f_n\}$ is a Cauchy sequence in the metric d_{L^p}, $1 \le p < \infty$, which means for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, for all $n, m \ge N$,

$$d_{L^p}(f_n, f_m) = \left(\int |f_n - f_m|^p dm\right)^{\frac{1}{p}} < \varepsilon.$$

Suppose if $\{f_n\}$ is not a Cauchy sequence in the measure m, it means there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$, s.t. for any $N \in \mathbb{N}$, there exist $n, m \ge N$,

$$m(\{x|\,|f_n(x) - f_m(x)| \ge \varepsilon_0\}) \ge \delta_0.$$

We denote

$$E = \{x|\,|f_n(x) - f_m(x)| \ge \varepsilon_0\},$$

then

$$\left(\int |f_n - f_m|^p dm\right)^{\frac{1}{p}} \ge \left(\int_E |f_n - f_m|^p dm\right)^{\frac{1}{p}} \ge \varepsilon_0 \left(\int_E dm\right)^{\frac{1}{p}} \ge \varepsilon_0 \delta_0^{\frac{1}{p}}.$$

It is a contradiction. So $\{f_n\}$ is a Cauchy sequence in the measure m.

III.

Proof. $f \in L^1$ means that

$$\int |f(x)| dm < \infty.$$

$g \in L^\infty$ means that

$$b_g = ess\sup|g(x)| = \inf\{b|\,|g(x)| \le b, a.e.\} < \infty,$$

that is to say, for any $\varepsilon > 0$, there exists $E \in \mathcal{M}$, s.t. for any $x \in E$, $|g(x)| \le b_g + \varepsilon$, and $m(\mathbb{R}\backslash E) = 0$. It is easy to see that $|fg|$ is measurable, so from the exercise 7, No.1, we can get

$$\begin{gathered}\int |fg|dm = \int_E |fg|dm + \int_{\mathbb{R}\backslash E} |fg|dm = \int_E |fg|dm \\ \le (b_g+\varepsilon)\int_E |f|dm \le (b_g+\varepsilon)\int |f|dm = (b_g+\varepsilon)d_{L^1}(f,0),\end{gathered}$$

which holds for any $\varepsilon > 0$, thus we finally obtain

$$\int |fg|dm \le ess\sup|g(x)| \cdot d_{L^1(f,0)} = d_{L^1}(f,0)d_{L^\infty}(g,0).$$

IV.

Proof. **AT.** Measurable functions $\{f_n\}$ converges to a measurable function f in the measure m, means that, for each $\varepsilon > 0$, and each $\delta > 0$, there exists $N \in \mathbb{N}$,

$$m(\{x|\,|f_n(x) - f(x)| \ge \varepsilon\}) < \delta$$

holds for all $n \ge N$.

Assume that $\{f_n\}$ is not a Cauchy sequence in the measure m, it means there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$, s.t. for any $N \in \mathbb{N}$, there exist $n, m \ge N$,

$$m(\{x|\,|f_n(x) - f_m(x)| \ge \varepsilon_0\}) \ge \delta_0.$$

We denote

$$E = \{x|\,|f_n(x) - f_m(x)| \ge \varepsilon_0\},$$

for any $x \in E$,

$$\begin{aligned}\varepsilon_0 &\le |f_n(x) - f_m(x)| \\ &\le |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &\le 2\max\{|f_n(x) - f(x)|,\ |f_m(x) - f(x)|\}.\end{aligned}$$

Denote $E_{1(n,m)} = \{x \in E|\,|f_n(x) - f(x)| \ge |f_m(x) - f(x)|\,\}$, and

$$E_{2(n,m)} = E\backslash E_{1(n,m)} = \{x \in E|\,|f_m(x) - f(x)| > |f_n(x) - f(x)|\,\},$$

then

$$|f_n(x) - f(x)| \ge \frac{\varepsilon_0}{2},\ \text{for } x \in E_{1(n,m)}$$

and

$$|f_m(x) - f(x)| \ge \frac{\varepsilon_0}{2},\ \text{for } x \in E_{2(n,m)}.$$

Since $E = E_{1(n,m)} \cup E_{2(n,m)}$, $m(E) = m(E_{1(n,m)}) + m(E_{2(n,m)}) \ge \delta_0$. From this, we know that for any $\delta > 0$, $m(E_{1(n,m)}) < \delta$ and $m(E_{2(n,m)}) < \delta$ can't hold simultaneously, which is contradictory to the fact that $\{f_n\}$ converges to a measurable function f in the measure m.

V.

Proof. Let $p = \infty$. $\{x_{n,k}\} \subset l^\infty$ is a Cauchy sequence, then choose $\varepsilon = 1$, there exists $N \in \mathbb{N}$, for all $m, n \geq N$,

$$d_{l^\infty}(\{x_{n,k}\}, \{x_{m,k}\}) = \sup_k |x_{n,k} - x_{m,k}| \leq 1.$$

For the N, since $\{x_{N,k}\}_{k=1}^\infty \in l^\infty$, according to the definition of space l^∞,

$$\sup_k |x_{N,k}| = M < \infty.$$

For those $n \geq N$,

$$\begin{aligned}\sup_k |x_{n,k}| &= \sup_k |x_{n,k} - x_{N,k} + x_{N,k}| \leq \sup_k(|x_{n,k} - x_{N,k}| + |x_{N,k}|) \\ &\leq \sup_k |x_{n,k} - x_{N,k}| + \sup_k |x_{N,k}| \leq 1 + M.\end{aligned}$$

And for those $n < N$, since $\{x_{n,k}\}_{k=1}^\infty \in l^\infty$, we also know that

$$\sup_k |x_{n,k}| = M_n < \infty, \quad n = 1, 2, \cdots, N-1.$$

Denote $C = \max\{M_1, M_2, \cdots, M_{N-1}, M\}$, then we can get

$$\sup_k |x_{n,k}| \leq C,$$

for all $n \in \mathbb{N}$.