Answers to Exercise 7

I.

Proof. Step 1. We consider simple function:

$$\phi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

$\{E_j\}_{j=1}^n$ are disjoint measurable sets, then

$$\int\limits_E \phi dm = \int \phi\chi_E dm = \sum_{j=1}^{n} a_j m(E_j \cap E) \le \sum_{j=1}^{n} a_j m(E) = 0,$$

now

$$\phi\chi_E(x) = \sum_{j=1}^{n} a_j \chi_{E_j \cap E}(x).$$

Step 2. $f : R \longrightarrow [0, \infty]$ is measurable function, then

$$\int\limits_E f dm = \sup\Big\{ \int\limits_E \phi dm \Big| \phi \text{ simple and } \phi \le f \Big\} = 0.$$

Step 3. From step 2, for any function $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$, we can get

$$\int\limits_E f dm = \int\limits_E f^+ dm - \int\limits_E f^- dm = 0.$$

II.

Proof. Denote $C_f = \inf\{b_1 | \, |f(x)| \le b_1, a.e.\}$, and $C_g = \inf\{b_2 | \, |g(x)| \le b_2, a.e.\}$. For any $\varepsilon > 0$, there exists $E_1 \in \mathcal{M}$, s.t. $|f(x)| \le C_f + \varepsilon$, for any $x \in E_1$, $m(\mathbb{R}\backslash E_1) = 0$, and there also exists $E_2 \in \mathcal{M}$, s.t. $|g(x)| \le C_g + \varepsilon$, for any $x \in E_2$, $m(\mathbb{R}\backslash E_2) = 0$.

Now by the $\triangle$-inequality, for any $x \in E_1 \bigcap E_2$,

$$\begin{aligned} |\alpha f(x) + \beta g(x)| &\le |\alpha||f(x)| + |\beta||g(x)| = |\alpha|(|f(x)| - \varepsilon) + |\beta|(|g(x)| - \varepsilon) + (|\alpha|\varepsilon + |\beta|\varepsilon) \\ &\le |\alpha|C_f + |\beta|C_g + (|\alpha| + |\beta|)\varepsilon, \end{aligned}$$

and since $m(\mathbb{R}\backslash(E_1 \cap E_2)) = m(\mathbb{R}\backslash E_1 \cup \mathbb{R}\backslash E_2) \le m(\mathbb{R}\backslash E_1) + m(\mathbb{R}\backslash E_2) = 0$, we take the infimum on both sides of the above inequality:

$$\begin{aligned} & \inf\{b | \, |\alpha f(x) + \beta g(x)| \le b, a.e.\} \\ \le \; & |\alpha| \inf\{b_1 | \, |f(x)| \le b_1, a.e.\} + |\beta| \inf\{b_2 | \, |g(x)| \le b_2, \, a.e.\} + \inf\{(|\alpha| + |\beta|)\varepsilon\} \\ = \; & |\alpha| \inf\{b_1 | \, |f(x)| \le b_1, a.e.\} + |\beta| \inf\{b_2 | \, |g(x)| \le b_2, \, a.e.\} \end{aligned}$$

III.

Proof. $d_{L^\infty} : L^\infty \times L^\infty \longrightarrow \mathbb{R}$, for any two functions f, g, $d_{L^\infty}(f, g) = ess \sup |f - g|$;

1. $d_{L^\infty}(f,g) = ess\sup|f-g| \geq 0$;
2. $d_{L^\infty}(f,g) = ess\sup|f-g| = \inf\{b|\,|f(x)-g(x)| \leq b, a.e.\} = 0 \Longleftrightarrow |f(x)-g(x)| = 0, a.e. \Longleftrightarrow f(x) = g(x), a.e.$;
3. $d_{L^\infty}(f,g) = ess\sup|f-g| = \inf\{b|\,|f(x)-g(x)| \leq b, a.e.\} = \inf\{b|\,|g(x)-f(x)| \leq b, a.e.\} = ess\sup|g-f| = d_{L^\infty}(g,f)$;
4. by question No.2 proved above, let $\alpha = 1$ and $\beta = 1$, for any other function $h \in L^\infty$, $d_{L^\infty}(f,g) = \inf\{b|\,|f-g| \leq b, a.e.\} = \inf\{b|\,|(f-h)+(h-g)| \leq b, a.e.\} \leq \inf\{b_1|\,|f-h| \leq b_1, a.e.\} + \inf\{b_2|\,|h-g| \leq b_2, a.e.\} = d_{L^\infty}(f,h) + d_{L^\infty}(h,g)$.

Thus, d_{L^∞} is indeed a metric.

IV.

Proof. If $ab = 0$, it is trivial; if $ab > 0$, let's consider function $f(x) = \log x$, $x > 0$. Since $f'(x) = \frac{1}{x}$ is decreasing, we know that $f(x) = \log x$ is concave function. In fact, suppose $0 < x < u < y$, apply mean value theorem for differentiation to find r and s with $0 < x < r < u < s < y$, such that

$$f'(r) = \frac{f(u)-f(x)}{u-x} = \frac{\log u - \log x}{u-x},$$

and

$$f'(s) = \frac{f(y)-f(u)}{y-u} = \frac{\log y - \log u}{y-u}.$$

Since $f'(x)$ is decreasing, $f'(r) \geq f'(s)$, this gives

$$\frac{\log u - \log x}{u-x} \geq \frac{\log y - \log u}{y-u},$$

whenever $0 < x < u < y$. In particular, by letting $u = (1-t)x + ty$, where $0 < t < 1$,

$$\frac{\log u - \log x}{t(y-x)} \geq \frac{\log y - \log u}{(1-t)(y-x)},$$

hence

$$(1-t)(\log u - \log x) \geq t(\log y - \log u),$$

put it in another way,

$$\log u = \log((1-t)x + ty) \geq (1-t)\log x + t\log y.$$

So, $\forall a$, $b \geq 0$, and $0 < \lambda < 1$,

$$\lambda \log a + (1-\lambda)\log b \leq \log(\lambda a + (1-\lambda)b),$$

which equals to

$$\log a^\lambda b^{1-\lambda} \leq \log(\lambda a + (1-\lambda)b).$$

Since $\log x$ itself is increasing, $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$.

V.

Proof. Suppose for a while $0 < M < +\infty$, otherwise is trivial. If $m(E) = +\infty$, the inequality holds; if $m(E) < +\infty$, take $\phi(x) = M\chi_E(x)$ a simple function, then $|f\chi_E| \leq \phi$. Further, $\int \phi dm = Mm(E) < +\infty$, by theorem 2.26 (c) and (b'), $|\int f\chi_E dm| \leq \int |f\chi_E| dm \leq \int \phi dm = Mm(E)$.