3 Classical and modern function spaces

The theories of function spaces are, in many cases, based on the measure and integration theory. In Analysis 4 we shall mostly restrict ourselves to $\mathbb{R}$, since the measure space $(\mathbb{R}, \mathcal{M}, m)$ is what we know best. We note that many of the function spaces below (see e.g. L_p spaces) could be considered in a more general setting than just in $\mathbb{R}$.

The linearity of a given function space is often proved by the classical Hölder and Minkowski inequalities. These inequalities have a great number of applications in various branches of analysis. Therefore, the first subsection is devoted to state and prove these two inequalities.

The remaining part of this section is devoted to function spaces. We shall begin from the classical L^p spaces and end up with Q_p spaces partly developed in the University of Joensuu.

Hölder and Minkowski inequalities

To prove the two classical inequalities, we need

Lemma 3.1 *If $a, b \geq 0$ and $0 < \lambda < 1$, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

with equality only if $a = b$.

Hölder Inequality: *Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.*[1] *Suppose that $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ and $g : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ are measurable functions such that $|f|^p$ and $|g|^q$ are integrable. Then $|fg|$ is integrable and, for any measurable set E, we have*

$$\int_E |fg|\, dm \leq \left(\int_E |f|^p\, dm \right)^{1/p} \left(\int_E |g|^q\, dm \right)^{1/q}. \tag{3.1}$$

Minkowski Inequality: *Let $1 \leq p < \infty$. Suppose that $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ and $g : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ are measurable functions such that $|f|^p$ and $|g|^p$ are integrable. Then $|f+g|^p$ is integrable and, for any measurable set E, we have*

$$\left(\int_E |f+g|^p\, dm \right)^{1/p} \leq \left(\int_E |f|^p\, dm \right)^{1/p} + \left(\int_E |g|^p\, dm \right)^{1/p}. \tag{3.2}$$

L^p spaces

Suppose that $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ is a measurable function and there exists $b > 0$ such that $f(x) \leq b$ a.e. Then we can define the *essential supremum* (*oleellinen supremum*) of f to be

$$\operatorname{ess\,sup} f = \inf\{b \mid f(x) \leq b \text{ a.e.}\}.$$

Further, f is said to be *essentially bounded* (*oleellisesti rajoitettu*) if there exists $b > 0$ such that $|f(x)| \leq b$ a.e.

[1] Such numbers p and q are called *conjucate indices* (*konjukaatti-indeksit*).

Definition 3.2 For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is the family of measurable functions $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ such that

$$\left(\int |f|^p \, dm \right)^{1/p} < \infty.$$

Further, $L^\infty(\mathbb{R})$ is the family of measurable functions $f : \mathbb{R} \longrightarrow \widehat{\mathbb{R}}$ such that

$$\text{ess sup} \, |f| < \infty.$$

For shortness, we denote $L^p = L^p(\mathbb{R})$ and $L^\infty = L^\infty(\mathbb{R})$.

Theorem 3.3 *The family L^1 is the family of all Lebesgue integrable functions.*

Theorem 3.4 *Let $1 \leq p \leq \infty$.*

(a) The space L^p is a linear vector space.

(b) Define $d_{L^p} : L^p \times L^p \longrightarrow \mathbb{R}_+$,

$$d_{L^p}(f, g) = \begin{cases} \left(\int |f - g|^p \, dm \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup} \, |f - g|, & p = \infty. \end{cases}$$

Considering two functions $f, g \in L^p$ to be equivalent if $f(x) = g(x)$ a.e., then d_{L^p} becomes a metric, that is, (L^p, d_{L^p}) is a metric space.

The metric d_{L^p} in the above theorem is called a *standard metric* on L^p and, unless otherwise stated, L^p will be assumed to have this metric. (Sometimes d_{L^p} is called a pseudo metric in the literature.)

Theorem 3.5 *The following two assertions hold:*

(a) If $f \in L^p$ and $g \in L^q$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1$.

(b) If $f \in L^p$ and $g \in L^\infty$, where $1 \leq p \leq \infty$, then $fg \in L^p$.

The remaining part of this subsection aims to show the completeness of the L^p spaces, that is, the Cauchy sequences in each L^p space converge. This result is usually referred to as *Riesz-Fischer Theorem*. To this end, some auxiliary results will be needed.

We have already defined pointwise and uniform convergence for a given sequence of functions. We still need *convergence in the measure m* (*suppeneminen mitan m suhteen*) and *convergence in the metric d_{L^p}* (*suppeneminen d_{L^p}-metriikan suhteen*).

Definition 3.6 A sequence $\{f_n\}$ of measurable functions is said to converge to f in the measure m if, for each $\varepsilon > 0$ and each $\delta > 0$ there exists an $N \in \mathbb{N}$ such that

$$m(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) < \delta$$

for all $n \geq N$.

Lemma 3.7 *If $\{f_n\}$ is a Cauchy sequence in the measure m, then there is a subsequence of $\{f_n\}$ which is a Cauchy sequence a.e.*

Definition 3.8 Given $1 \leq p \leq \infty$, let $\{f_n\}$ be a sequence of functions in L^p and let $f \in L^p$. We say that $\{f_n\}$ converges to f in the metric d_{L^p}, if

$$\lim_{n \to \infty} d_{L^p}(f_n, f) = 0.$$

Lemma 3.9 *If $\{f_n\}$ is a Cauchy sequence in the metric d_{L^p}, $1 \leq p < \infty$, then $\{f_n\}$ is a Cauchy sequence in the measure m.*

Finally we are ready for

Riesz-Fischer Theorem: *Let $1 \leq p \leq \infty$ and let $\{f_n\}$ be a Cauchy sequence in the metric d_{L^p}. Then $\{f_n\}$ converges to some $f \in L^p$ in the metric d_{L^p}.*

Remark. Riesz-Fischer Theorem along with Theorem 3.4 say that each L^p space is a complete metric space.

The following result shows us that L^p spaces defined in a set of finite measure satisfy the nesting property.

Theorem 3.10 *Let $1 \leq p < q \leq \infty$ and let $a, b \in \mathbb{R}$ be such that $a < b$. Then $L^q[a,b] \subset L^p[a,b]$.*

ℓ^p spaces

Recall from Section 1 that sequences can be understood as discrete functions. Therefore, ℓ^p spaces in the following definition can be regarded as a discrete version of L^p spaces.

Definition 3.11 For $1 \leq p < \infty$, $\ell^p(\mathbb{R})$ (resp. $\ell^p(\mathbb{C})$) is the family of all sequences $\{x_n\}$ in $\mathbb{R}$ (resp. in $\mathbb{C}$) such that

$$\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

Further, $\ell^\infty(\mathbb{R})$ (resp. $\ell^p(\mathbb{C})$) is the family of all sequences $\{x_n\}$ in $\mathbb{R}$ (resp. in $\mathbb{C}$) such that

$$\sup_n |x_n| < \infty.$$

For shortness, we denote $\ell^p = \ell^p(\mathbb{R})$ and $\ell^\infty = \ell^\infty(\mathbb{R})$.

Remarks. (1) Recall that real sequences are countable and therefore measurable by Corollary 2.10.

(2) Analogously to L^p spaces (see Theorem 3.3), we call a sequence $\{x_n\}$ integrable if and only if $\{x_n\} \in \ell^1$. The integral of $\{x_n\}$ is simply the sum $\sum_{n=1}^{\infty} x_n$. This fact is based on a concept of a counting measure below.

Definition 3.12 Let $\mathcal{M}_c$ be the family of all subsets of $\mathbb{N}$ and, for any $A \subset \mathbb{N}$, define $m_c(A)$ be the number of elements in A. Then the triple $(\mathbb{N}, \mathcal{M}_c, m_c)$ is a measure space and m_c is called a counting measure (lukumäärämitta).

Theorem 3.13 *If f is a sequence $\{x_n\}$ in $\mathbb{R}$, that is, if $f : \mathbb{N} \longrightarrow \mathbb{R}$ is such that $f(n) = x_n$, then*

$$\int f \, dm_c = \sum_n x_n, \tag{3.3}$$

provided the series in (3.3) exists. Here, $\int f \, dm_c = \int_{\mathbb{N}} f \, dm_c$.

Hölder and Minkowski inequalities hold in fact for any measure and not just for the Lebesgue measure. Therefore, we apply the two inequalities in the case of a counting measure and make use of Theorem 3.13 to get:

Hölder Inequality (for series): *Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{a_n\} \in \ell^p$ and $\{b_n\} \in \ell^q$. Then $\{a_n b_n\} \in \ell^1$ and*

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}. \tag{3.4}$$

Minkowski Inequality (for series): *Let $1 \leq p < \infty$. Suppose that $\{a_n\} \in \ell^p$ and $\{b_n\} \in \ell^p$. Then $\{a_n + b_n\} \in \ell^p$ and*

$$\left(\sum_{n=1}^{\infty} |a_n + b_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |b_n|^p \right)^{1/p}. \tag{3.5}$$

Theorem 3.14 *Let $1 \leq p \leq \infty$.*

(a) The space ℓ^p is a linear vector space.

(b) Define $d_{\ell^p} : \ell^p \times \ell^p \longrightarrow \mathbb{R}_+$,

$$d_{\ell^p}(\{a_n\}, \{b_n\}) = \begin{cases} \left(\sum_n |a_n - b_n|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_n |a_n - b_n|, & p = \infty. \end{cases}$$

Then d_{ℓ^p} is a metric, that is, (ℓ^p, d_{ℓ^p}) is a metric space.

We aim to prove the completeness of the ℓ^p spaces. To this end, we need to know what is meant by a Cauchy sequence in ℓ^p. Denote

$$\{x_{n,k}\} = \{\{x_{1,k}\}_{k=1}^{\infty}, \{x_{2,k}\}_{k=1}^{\infty}, \ldots\},$$

where, for each $n \in \mathbb{N}$, $\{x_{n,k}\}_{k=1}^{\infty} \in \ell^p$. Then $\{x_{n,k}\}$ is a Cauchy sequence in ℓ^p, provided that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$d_{\ell^p}(\{x_{n,k}\}, \{x_{m,k}\}) < \varepsilon,$$

as $n, m \geq N$.

We shall make use of

Lemma 3.15 *Let $1 \leq p \leq \infty$. If $\{x_{n,k}\} \subset \ell^p$ is a Cauchy sequence, then there exists a uniform constant $C > 0$ such that*

$$\left(\sum_{k=1}^{\infty} |x_{n,k}|^p \right)^{1/p} \leq C \quad (p < \infty) \quad \text{or} \quad \sup_{k \in \mathbb{N}} |x_{n,k}| \leq C \quad (p = \infty),$$

for all $n \in \mathbb{N}$.

As for the L^p spaces, we have

Theorem 3.16 *For every $1 \leq p \leq \infty$, every Cauchy sequence in ℓ^p converges to a sequence in ℓ^p in the metric d_{ℓ^p}.*

Complex function spaces

In this subsection we introduce some complex function spaces. These spaces are spaces of functions f analytic in the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

Roughly speaking, analytic functions f in D are those functions $f : D \longrightarrow \mathbb{C}$ for which the complex derivative $\frac{d}{dz} f = f'$ exists.

All the spaces mentioned below are really complete linear spaces (proofs will be omitted).

To begin with, we mention an analytic correspondence to L^p spaces known as the *Hardy spaces* H^p. Let $1 \leq p < \infty$. A function f analytic in D is said to belong to H^p, if

$$\sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.$$

Further, f is said to belong to H^∞, if

$$\sup_{z \in D} |f(z)| < \infty.$$

A function f analytic in D is said to belong to the *Dirichlet space* $\mathcal{D}$, if

$$\int\int_D |f'(z)|^2 \, dm(z) < \infty, \tag{3.6}$$

where $z = re^{i\theta}$ and $dm(z) = rdrd\theta$ is the areal Lebesgue measure in D. The integral in (3.6) is the area of the image of D under f counting multiplicities. Therefore, for every function in the Diriclet space, the image area (counting multiplicities) is bounded.

A function f analytic in D is said to belong to the *Bloch space*, if

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$

The image area of any Bloch function do not contain arbitrary large schlicht discs (Bloch-funktioiden kuvajoukot eivät sisällä mielivaltaisen suuria sileästi/yksiarvoisesti kuvautuvia kiekkoja).

Following R. Aulaskari, J. Xiao and R. Zhao (1995), we define the Q_p *spaces* as follows. Let $0 \leq p < \infty$. A function f analytic in D is said to belong to Q_p, if

$$\sup_{a \in D} \int\int_D |f'(z)|^2 g(z,a)^p \, dm(z) < \infty,$$

where $dm(z) = rdrd\theta$ is the areal Lebesgue measure in D and

$$g(z,a) = \log \left| \frac{1 - \bar{a}z}{z - a} \right|$$

is the Green's function in D with logarithmic singularity at a. We note that $Q_0 = \mathcal{D}$, $Q_1 = BMOA$ and, for any $p > 1$, $Q_p = \mathcal{B}$. Further, the Q_p spaces satisfy the following strict nesting property: If $0 < p_1 < p_2 < 1$, then

$$\mathcal{D} \subsetneq Q_{p_1} \subsetneq Q_{p_2} \subsetneq BMOA \subsetneq \mathcal{B}.$$