5 Inner product spaces

The previous section introduced the concept of the norm of a vector as a generalization of the idea of the length of a vector in $\mathbb{R}^3$. However, the length of a vector in $\mathbb{R}^3$ is not the only geometric concept which can be expressed algebraically. Namely, if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ then the angle between them, say θ, can be obtained by using the scalar product

$$< x, y >= x_1y_1 + x_2y_2 + x_3y_3 = ||x||\,||y|| \cos\theta,$$

where

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{< x, x >}$$

and

$$||y|| = \sqrt{y_1^2 + y_2^2 + y_3^2} = \sqrt{< y, y >}$$

are the lengths of x and y, respectively.

The scalar product in $\mathbb{R}^3$ is such a useful concept that we would like to extend it to other spaces — this is essentially what will be done in the first subsection. Some of the function spaces known so far appear to possess inner products and hence being inner product spaces of functions.

Real and complex inner products

We will see that it is necessary to distinguish between real and complex spaces.

Definition 5.1 Let X be a real vector space. An inner product on X is a function $< \cdot, \cdot >: X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{R}$,

(a) $< x, x >\geq 0$,

(b) $< x, x >= 0$ if and only if $x = 0$,

(c) $< \alpha x + \beta y, z >= \alpha < x, z > +\beta < y, z >$,

(d) $< x, y >=< y, x >$.

Example. The function $< x, y >: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $< x, y >= \sum_{j=1}^{n} x_j y_j$ is an inner product on $\mathbb{R}^n$ and is called the *standard inner product* on $\mathbb{R}^n$.

Definition 5.2 Let X be a complex vector space. An inner product on X is a function $< \cdot, \cdot >: X \times X \longrightarrow \mathbb{C}$ such that for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{C}$,

(a) $< x, x >\in \mathbb{R}$ and $< x, x >\geq 0$,

(b) $< x, x >= 0$ if and only if $x = 0$,

(c) $< \alpha x + \beta y, z >= \alpha < x, z > +\beta < y, z >$,

(d) $< x, y >= \overline{< y, x >}$.

Example. The function $< x, y >: \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $< x, y >= \sum_{j=1}^{n} x_j \bar{y}_j$ is an inner product on $\mathbb{C}^n$ and is called the *standard inner product* on $\mathbb{C}^n$.

Some inner product spaces

Definition 5.3 A real or complex vector space X with an inner product $< \cdot, \cdot >$ is called an inner product space.

In general, an inner product can be defined on any finite-dimensional vector space, as seen in the following:

Example. Let X be a finite-dimensional vector space with linearly independent basis $\{e_1, \ldots, e_n\}$. Let $x, y \in X$ have the (unique) representations $x = \sum_{j=1}^{n} \lambda_j e_j$ and $y = \sum_{j=1}^{n} \mu_j e_j$. The function $< \cdot, \cdot >: X \times X \longrightarrow \mathbb{F}$ defined by $< x, y >= \displaystyle\sum_{j=1}^{n} \lambda_j \bar{\mu}_j$ is an inner product on X.

Clearly, the inner product in the example above depends on the basis chosen and so we only obtain a "standard" inner product when there is some natural "standard" basis for the space.

We next show that L^2 and ℓ^2 are inner product spaces (standard examples).

Example. If $f, g \in L^2$ then $fg \in L^1$ and the function $< \cdot, \cdot >: L^2 \times L^2 \longrightarrow \mathbb{R}$ defined by $< f, g >= \int fg\, dm$ is an inner product on L^2 called the *standard inner product* on L^2.

Example. If $a = \{a_n\}, b = \{b_n\} \in \ell^2$, then $\{a_n b_n\} \in \ell^1$ and the function $< \cdot, \cdot >: \ell^2 \times \ell^2 \longrightarrow \mathbb{R}$ defined by $< a, b >= \sum_{n=1}^{\infty} a_n b_n$ is an inner product on ℓ^2 called the *standard inner product* on ℓ^2.

Remark. The two examples above naturally hold in a more general setting:

(1) If (X, Σ, μ) is any measure space and and $f, g \in L^2(X)$, then $f\bar{g} \in L^1(X)$ and the function $< \cdot, \cdot >: L^2(X) \times L^2(X) \longrightarrow \mathbb{F}$ defined by $< f, g >= \int_X f\bar{g}\, d\mu$ is an inner on $L^2(X)$.

(2) If $a = \{a_n\}, b = \{b_n\} \in \ell^2(F)$, then $\{a_n b_n\} \in \ell^1(F)$ and the function $< \cdot, \cdot >: \ell^2 \times \ell^2 \longrightarrow \mathbb{F}$ defined by $< a, b >= \sum_{n=1}^{\infty} a_n \bar{b}_n$ is an inner product on $\ell^2(F)$.

Example. Let X and Y be inner product spaces with inner products $< \cdot, \cdot >_1$ and $< \cdot, \cdot >_2$, respectively, and let $Z = X \times Y$. Then the function $< \cdot, \cdot >: Z \times Z \longrightarrow \mathbb{F}$ defined by $< (x, y), (u, v) >=< x, u >_1 + < y, v >_2$ is an inner product on Z.

Some elementary algebraic identities

We prove some elementary algebraic identities satisfied by inner products. Our first result does not seem important, but we shall make use of it below.

Lemma 5.4 *Let X be an inner product space, $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$. Then*

(a) $< 0, y >=< x, 0 >= 0$,

(b) $< x, \alpha y + \beta z >= \bar{\alpha} < x, y > + \bar{\beta} < x, z >$,

(c) $<\alpha x + \beta y, \alpha x + \beta y>=$
$|\alpha|^2 <x, x> + \alpha\bar{\beta} <x, y> + \beta\bar{\alpha} <y, x> + |\beta|^2 <y, y>$.

In the introduction to this section we noted that if $x \in \mathbb{R}^3$ and $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^3$, then $\sqrt{< x, x >}$ gives the usual Euclidean length, or norm, of x. We next show that for a general inner product space X the same expression defines a norm on X.

Lemma 5.5 *Let X be an inner product space and let $x, y \in X$. Then*

(a) $| < x, y > |^2 \leq < x, x >< y, y >$,

(b) the function $|| \cdot || : X \longrightarrow \mathbb{R}$ defined by $||x|| = \sqrt{< x, x >}$ is a norm on X.

The norm $||x|| = \sqrt{< x, x >}$ defined in Lemma 5.5 on the inner product space X is said to be *induced* by the inner product $< \cdot, \cdot >$. The lemma shows that, by using the induced norm, every inner product space can be regarded as a normed space. Therefore, whenever using a norm on an inner product space X, it is customary to use the induced norm without specifically mentioning it each time. With this convention the inequality in Lemma 5.5(a) can be rewritten as

$$| < x, y > | \leq ||x|| \, ||y||. \tag{5.1}$$

Inequality (5.1) is known as the *Cauchy-Schwarz inequality.*

Lemma 5.6 *Let X be an inner product space with inner product $< \cdot, \cdot >$. Then for all $u, v, x, y \in X$,*

(a) $< u + v, x + y > - < u - v, x - y >= 2 < u, y > +2 < v, x >$,

(b) for complex X,

$$\begin{aligned} 4 < u, y > = \; & < u + v, x + y > - < u - v, x - y > \\ & + i < u + iv, x + iy > - i < u - iv, x - iy > . \end{aligned}$$

Theorem 5.7 *Let X be an inner product space with inner product $< \cdot, \cdot >$ and induced norm $|| \cdot ||$. Then for all $x, y \in X$,*

(a) the parallelogram rule:

$$||x + y||^2 + ||x - y||^2 = 2 \left(||x||^2 + ||y||^2 \right),$$

(b) for real X, $4 < x, y >= ||x + y||^2 - ||x - y||^2$,

(c) for complex X, the polarization identity:

$$4 < x, y >= ||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2.$$

Since every inner product space has an induced norm, a natural question is whether *every* norm is induced by an inner product. The answer is no. One way to show that a given norm on a vector space is not induced by an inner product is to show that it does not satisfy the parallelogram rule.

Example. The standard norm on the space $C_{\mathbb{R}}[0, 1]$ is not induced by an inner product.

Orthogonality

The main reason we introduced inner products was in the hope of extending the concept of angles between vectors. From the Cauchy-Schwarz inequality (5.1) for *real* inner product spaces, if x and y are non-zero vectors, then

$$-1 \leq \frac{< x, y >}{||x||\,||y||} \leq 1,$$

and so the angle between x and y can be defined to be

$$\theta = \cos^{-1}\left(\frac{< x, y >}{||x||\,||y||}\right).$$

For complex inner product spaces the situation is more difficult since the inner product $< x, y >$ may be complex and it is not clear what a complex "angle" would mean. However, an important special case can be considered, namely when $< x, y >= 0$:

Definition 5.8 Let X be an inner product space. The vectors $x, y \in X$ are said to be orthogonal if $< x, y >= 0$.

Example. Recall the orthogonality in $\mathbb{R}^3$: If $x, y \in \mathbb{R}^3 \setminus \{0\}$ are such that

$$0 =< x, y >= x_1y_1 + x_2y_2 + x_3y_3 = ||x||\,||y|| \cos\theta,$$

then $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$.

From the standard linear algebra we are familiar with the concept of orthonormal sets of vectors in finite dimensional real inner product spaces. This concept can be extended to arbitrary inner product spaces.

Definition 5.9 Let X be an inner product space. The set $\{e_1, \ldots, e_n\} \subset X$ is said to be orthonormal if $||e_k|| = 1$ for $1 \leq k \leq n$ and $< e_j, e_k >= 0$ for all $1 \leq j, k \leq n$.

Now the theories of orthonormal basis', orthogonal subspaces etc. can be developed similarly as in the standard linear algebra course.

Hilbert spaces

As completeness is an important property in normed spaces, this is also true for inner product spaces. Complete inner product were called Banach spaces, and complete inner product spaces are *Hilbert spaces*:

Definition 5.10 An inner product space which is complete with respect to the metric associated with the norm induced by the inner product is called a Hilbert space.

Theorem 5.11 *The following inner product spaces are Hilbert spaces:*

(a) L^2 *with the standard inner product,*

(b) ℓ^2 *with the standard inner product,*

(c) all finite-dimensional inner product spaces.

There are, of course, many other Hilbert spaces than those mentioned in Theorem 5.11.