COMPLEX ANALYSIS 2

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Background

This material of Complex Analysis 2 assumes that the reader is familiar with basic facts of complex analysis. In particular, the reader should be able to understand (and work with) complex number including their polar representation, and elementary complex functions such as the exponential function as well as basic trigonometric functions. Of course, the reader should also know the notion of analytic functions as well as Cauchy–Riemann equations, Möbius transformations, power series and complex integration. In particular, we shall also apply Cauchy integral theorem, Cauchy integral formula, power series representation of analytic function, Gauss mean value theorem, Cauchy inequalities, elementary uniqueness theorem of analytic functions, maximum principle and the Schwarz lemma, whenever needed.

1. Singularities for analytic functions

Unless otherwise specified, we are considering analytic functions in domains in question.

Definition. Given f, $z = a$ is an *isolated* singularity of f, if there exists $R > 0$ such that f is analytic in $0 < |z - a| < R$. The point $z = a$ is a *removable* singularity, if there exists an analytic $g\colon B(a, R) \to \mathbb{C}$ such that $g(z) = f(z)$ for all z such that $0 < |z - a| < R$.

Theorem 1.2. *A singularity at $z = a$ is removable if and only if*

$$\lim_{z \to a} (z - a) f(z) = 0.$$

Proof. (1) As an analytic function, g is continuous, hence bounded around a. Therefore,

$$\lim_{\substack{z \to a \\ z \neq a}} (z - a) f(z) = \lim_{\substack{z \to a \\ z \neq a}} (z - a) g(z) = 0$$

trivially.

(2) Let us define $h\colon B(a, R) \to \mathbb{C}$ by

$$h(z) := \begin{cases} (z - a) f(z), & z \neq a \\ 0, & z = a. \end{cases}$$

Clearly, h is continuous. We first prove that h is analytic. By the Cauchy integral theorem,

$$\int_\gamma h(\zeta)\, d\zeta = 0,$$

provided γ is a piecewise continuously differentiable closed path in $B(a, R)$. This implies the existence of $H\colon B(a, r) \to \mathbb{C}$ such that $H' = h$. Clearly, H is analytic. Therefore, H is infinitely differentiable, and so $h = H'$ also is differentiable and therefore analytic in $B(a, R)$. This implies that h can be represented as

$$h(z) = \sum_{j=0}^{\infty} a_j (z - a)^j.$$

Since $h(a) = 0$,

$$h(z) = \sum_{j=1}^{\infty} a_j (z - a)^j = (z - a) \sum_{j=0}^{\infty} a_{j+1} (z - a)^j.$$

As a convergent power series, $\sum_{j=0}^{\infty} a_{j+1}(z - a)^j =: g(z)$ determines an analytic function in $B(a, R)$. If $z \neq a$, then

$$(z - a)f(z) = h(z) = (z - a)g(z),$$

and so $f(z) = g(z)$. □

Definition 1.3. An isolated singularity $z = a$ is a *pole*, if $\lim_{z\to a} |f(z)| = \infty$. If an isolated singularity is neither removable nor a pole, then it is called an *essential* singularity.

Theorem 1.4. *For a pole $z = a$ of f, there exists $m \in \mathbb{N}$ and an analytic function $g\colon B(a, R) \to \mathbb{C}$ such that*

$$f(z) = (z - a)^{-m} g(z)$$

for any $0 < |z - a| < R$.

Proof. Since $\lim_{z\to a} \frac{1}{|f(z)|} = 0$, we have

$$\lim_{z\to a} (z - a) \frac{1}{f(z)} = 0.$$

By Theorem 1.2, $z = a$ is a removable singularity for $\frac{1}{f(z)}$. Therefore, there exists an analytic $h\colon B(a, R) \to \mathbb{C}$ such that

$$h(z) = \frac{1}{f(z)} \qquad \text{for all } 0 < |z - a| < R.$$

By the power series representation,

$$\begin{aligned} h(z) &= \sum_{j=m}^{\infty} a_j (z - a)^j = (z - a)^m \sum_{j=0}^{\infty} a_{m+j} (z - a)^j \\ &= (z - a)^m h_1(z), \end{aligned}$$

where $m \in \mathbb{N}$, h_1 is analytic in $B(a,R)$ and $h_1(a) \neq 0$. Since

$$\frac{1}{f(z)} = (z-a)^m h_1(z), \qquad 0 < |z-a| < R,$$

we get

$$(z-a)^m f(z) = \big(h_1(z)\big)^{-1} \tag{1.1}$$

Since $0 < |h_1(a)| < \infty$, it follows that $\frac{1}{h_1(z)}$ is bounded around $z = a$ and so

$$\lim_{z \to a} (z-a) \frac{1}{h_1(z)} = 0.$$

Therefore, $\frac{1}{h_1}$ has a removable singularity at $z = a$ and so there exists an analytic $g \colon B(a,R) \to \mathbb{C}$ so that $g(z) = \frac{1}{h_1(z)}$ for $0 < |z-a| < R$. By (1.1),

$$f(z) = (z-a)^{-m} g(z), \qquad 0 < |z-a| < R. \quad \square$$

Definition 1.5. Assume f has a pole at $z = a$. The smallest integer $m \in \mathbb{N}$ such that $(z-a)^m f(z)$ has a removable singularity at $z = a$, is the *multiplicity* of the pole.

Exercise 1.1. Consider the following functions around $z = 0$:

(1) $f(z) = \frac{1}{z}$
(2) $f(z) = \frac{\sin z}{z}$
(3) $f(z) = \frac{\cos z}{z}$
(4) $f(z) = \frac{1}{1-e^z}$
(5) $f(z) = e^{1/z}$
(6) $f(z) = z \sin \frac{1}{z}$.

Determine whether $z = 0$ is removable, a pole or an essential singularity. In case of a pole, determine also the multiplicity.

Theorem 1.6. (Laurent series). *A function f analytic in an annulus $0 \le R_1 < |z-a| < R_2 \le \infty$ admits a unique representation*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j.$$

The series on the right hand side converges absolutely and uniformly in every annulus $r_1 < |z-a| < r_2$ such that $R_1 < r_1 <_2 < R_2$. The coefficients a_j are determined by

$$a_j := \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{j+1}} \, d\zeta \tag{$\sim$}$$

where $\gamma_r := \{\, |z-a| = r \,\}$, $R_1 < r < R_2$.

Proof. Omitted, see

Theorem 1.7. *Let $z = a$ be an isolated singularity of f and*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j$$

be its Laurent series expansion in $0 < |z-a| < R$. Then

(1) *$z = a$ is removable if and only if $a_j = 0$ for $j \leq -1$,*
(2) *$z = a$ is a pole of multiplicity $m \in \mathbb{N}$ if and only if $a_{-m} \neq 0$ and $a_j = 0$ for $j \leq -(m+1)$,*
(3) *$z = a$ is essential if and only if $a_j \neq 0$ for infinitely many negative integers j.*

Exercise 1.2. Prove Theorem 1.7.

Theorem 1.8. (Casorati–Weierstraß). *If f has an essential singularity at $z = a$, then for every $\delta > 0$,*

$$\overline{f\big(B(a,\delta) \setminus \{a\}\big)} = \mathbb{C}.$$

Proof. We have to prove: Given $c \in \mathbb{C}$ and $\varepsilon > 0$, there exists for each $\delta > 0$ a point $z \neq a$ such that $|z-a| < \delta$ and $|f(z) - c| < \varepsilon$. If this is not the case, then there exists $c \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - c| \geq \varepsilon$ for all $z \in B(a,\delta)$, $z \neq a$. But then

$$\lim_{\substack{z \to a \\ z \neq a}} \left| \frac{f(z) - c}{z - a} \right| = \infty.$$

This means that $\frac{f(z)-c}{z-a}$ has a pole at $z = a$. Let m be the multiplicity. Then $m \geq 1$ and

$$g(z) := (z-a)^m \frac{f(z) - c}{z - a}$$

has a removable singularity. Therefore

$$0 = \lim_{z \to a} (z-a) g(z) = \lim_{z \to a} (z-a)^m \big(f(z) - c\big).$$

Then

$$\lim_{z \to a} (z-a)^m f(z) = \lim_{z \to a} \big[(z-a)^m \big(f(z) - c\big) + c(z-a)^m\big] = 0$$

and so

$$\lim_{z \to a} (z-a) \big(f(z)(z-a)^{m-1}\big) = 0.$$

Hence,

$$f(z)(z-a)^{m-1}$$

has a removable singularity at $z = a$. By Definition 1.1, there exists an analytic $g\colon B(a,\delta) \to \mathbb{C}$ such that

$$f(z) = \frac{g(z)}{(z-a)^{m-1}}, \qquad 0 < |z-a| < \delta.$$

If $m > 1$, then $\lim_{z \to a} |f(z)| = \infty$, hence f has a pole at $z = a$, and if $m = 1$, then $f(z)$ has a removable singularity at $z = a$. Both cases contradict the assumption of an essential singularity at $z = a$. □