The Cartan lemma is a purely geometric result addressing the geometry of a finite point set in the complex plane, having a number of applications into the analysis of canonical products.

Lemma 9.1. *Let $z_1, \ldots, z_n$ be given points in $\mathbb{C}$ and $H > 0$ be given. Then there exists closed disks $\Delta_1, \ldots, \Delta_m$, $m \leq n$, such that the sum of the radii of the disks $\Delta_1, \ldots, \Delta_m$ is $\leq 2H$ and that*

$$|z - z_1||z - z_2| \ldots |z - z_n| > (H/e)^n,$$

whenever $z \notin \bigcup_{j=1}^{m} \Delta_j$.

Remark. The points z_j in the assertion above are not necessarily distinct.

Proof. (1) Suppose first that there exists a disk Δ of radius H such that $\{z_1, \ldots, z_n\} \subset H$. Let now Δ_1 denote the disk of radius $2H$, with the same centre as Δ. Consider now any point $z \notin \Delta_1$. Then $|z - z_j| > H$ for each z_j, $j = 1, \ldots, n$. Therefore we obtain

$$|z - z_1||z - z_2| \cdots |z - z_m| > H^n > (H/e)^n.$$

(2) We now define k_1 to be the greatest natural number which satisfies the following condition: There exists a closed disk Δ_1' of radius $k_1 H/n$ such that at least k_1 points z_j are contained in this disk. Obviously, we must have $1 \leq k_1 < n$, the last inequality following as we don't have the case of the first part of the proof. Actually, Δ_1' contains exactly k_1 points z_j. In fact, if not, then Δ_1' contains at least $k_1 + 1$ points z_j. Then the disk of radius $(k_1 + 1)H/n$ with the same centre as Δ_1' results in a contradiction to the definition of k_1.

Renumbering now, if needed, we may assume that $z_1, \ldots, z_{k_1} \in \Delta_1'$ while $z_{k_1+1}, \ldots, z_n \notin \Delta_1'$. We now start repeating the process. So, let k_2 be the greatest natural number such that for a closed disk Δ_2' of radius $k_2 H/n$ at least (actually, exactly) k_2 points of $z_{k_1+1}, \ldots, z_n$ are contained in Δ_2'. Then we have $k_2 \leq k_1$; in fact, otherwise we would have a contradiction to the choice of k_1. We now repeat this process m times, $m \leq n$, so that all points $z_1, \ldots, z_n$ are contained in $\bigcup_{j=1}^{m} \Delta_j'$. Clearly, the disk Δ_j' ha radius $k_j H/n$ and $k_1 \geq k_2 \geq \cdots \geq k_n$. Since each Δ_j' contains exactly k_j points of $z_1, \ldots, z_n$, we must have $k_1 + k_2 + \cdots + k_m = n$. Therefore, the sum of their radii is

$$\frac{k_1}{n} H + \cdots + \frac{k_m}{n} H = \frac{k_1 + \cdots + k_m}{n} H = H.$$

Expand now the disks Δ_j', $j = 1, \ldots, n$, concentrically to Δ_j of radius $2\frac{k_j}{n} H$. Hence, the sum of the radii of the disks Δ_j is $= 2H$.

Consider now an arbitrary point $z \notin \bigcup_{j=1}^{m} \Delta_j$. Keep z fixed in what follows. We may assume, by renumbering the points $z_1, \ldots, z_n$ again, if needed, that

$$|z - z_1| \leq |z - z_2| \leq \cdots \leq |z - z_n|.$$

Assuming now that we have been able to prove that

$$|z - z_j| > \frac{j}{n} H, \qquad j = 1, \dots, n, \tag{10.1}$$

we obtain

$$\prod_{j=1}^{n} |z - z_j| > \prod_{j=1}^{n} \frac{j}{n} H = \frac{n!}{n^n} H^n \geq e^{-n} H^n = (H/e)^n.$$

In fact, this is an immediate consequence of

$$e^n = \sum_{j=0}^{\infty} \frac{1}{j!} n^j \geq \frac{1}{n!} n^n.$$

It remains to prove (10.1). We proceed to a contradiction by assuming that there exists at least one j such that $|z - z_j| \leq \frac{j}{n} H$. Let now p be the greatest natural number such that $k_p \geq j$. Such a number p exists. In fact, by monotonicity of the distances $|z - z_j|$, the disk of radius $\frac{j}{n} H$, centred at z, contains at least the points $z_1, \dots, z_j$, and so $k_1 \geq j$. Consider now the pairs of natural numbers (s, q) such that $s \leq j$, $q \leq p$.

We first proceed to prove that $z_s \notin \Delta'_q$. In fact, suppose for a while that we have $z_s \in \Delta'_q$ for some (s, q) such that $s \leq j$, $q \leq p$. By the definition of p, we have $k_q \geq j$. The radius of Δ'_q equals to $\frac{k_q}{n} H$ and Δ'_q contains k_q points of $z_1, \dots, z_n$. Let ζ be the centre of Δ'_q. Then

$$|z - \zeta| \leq |z - z_s| + |\zeta - z_s| \leq |z - z_j| + |\zeta - z_s| \leq \frac{j}{n} H + \frac{k_q}{n} H \leq 2 \frac{k_q}{n} H.$$

Therefore, we have $z \in \Delta_q$, contradicting to $z \notin \bigcup_{j=1}^{m} \Delta_j$.

Therefore, we have $z_s \notin \Delta'_q$ for all pairs (s, q) such that $s \leq j$, $q \leq p$. In particular, this means that

$$\{z_1, \dots, z_j\} \subset (\mathbb{C} \setminus \Delta'_p) \cap \dots \cap (\mathbb{C} \setminus \Delta'_1).$$

Since now

$$|z - z_1| \leq |z - z_2| \leq \dots \leq |z - z_j| \leq \frac{j}{n} H,$$

the disk of radius $\frac{j}{n} H$, centred at z, contains the points $z_1, \dots, z_j$. By the definition of k_{p+1}, which takes into account points of $z_1, \dots, z_n$, which are outside of $\bigcup_{j=1}^{p} \Delta'_j$, this means that $k_{p+1} \geq j$, a contradiction to the definition of p as the greatest number such that $k_p \geq j$. Therefore, (10.1) holds and we are done.