Recall first the definitions of the Weierstraß factors in Chapter 5:

$$\begin{cases} E_0(z) := 1 - z \\ E_\nu(z) := (1-z)e^{Q_\nu(z)} = (1-z)e^{z+\frac{1}{2}z^2+\cdots+\frac{1}{\nu}z^\nu}, \quad \nu \geq 1, \end{cases}$$

and the notion of the convergence in Chapter 9.

Let $f(z)$ now be an entire function of finite order ρ, and let $(z_n)_{n\in\mathbb{N}}$ be the sequence of its non-zero zeros, arranged according to increasing moduli. Let λ be the convergence exponent of $f(z)$ and define

$$\nu := \begin{cases} [\lambda] = & \text{the integer part of } \lambda, \text{ if } \lambda \text{ is non-integer} \\ \lambda - 1, & \text{if } \lambda \text{ is an integer and } \sum |z_j|^{-\lambda} \text{ converges} \\ \lambda & \text{otherwise.} \end{cases}$$

By Definition 9.2, $\sum |z_j|^{-(\nu+1)}$ converges, and

$$Q(z) = \prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right) \tag{11.1}$$

is an entire function with zeros exactly at (z_n). Therefore, $\lambda(Q) = \lambda$. By Theorem 9.9, $\lambda \leq \rho(Q)$.

The infinite product (11.1) is called the *canonical product determined by* (the non-zero zeros) of $f(z)$. Adding a suitable power z^m as an extra factor to $Q(z)$, we may take into account all zeros of $f(z)$.

Theorem 11.1. *For a canonical product,* $\lambda(Q) = \lambda = \rho(Q)$.

Proof. It suffices to prove that $\rho(Q) \leq \lambda$. To this end, we have to find a suitable majorant of $M(r, Q)$. Fix now z, $|z| = r$, and $\varepsilon > 0$. Obviously,

$$\log M(r, Q) = \log \max_{|z|=r} |Q(z)| = \max_{|z|=r} \log |Q(z)|.$$

Clearly,

$$\begin{aligned} \log |Q(z)| &= \log \prod_{j=1}^{\infty} \left| E_\nu\left(\frac{z}{z_j}\right)\right| \leq \sum_{|z/z_j|\geq 1/2} \log \left| E_\nu\left(\frac{z}{z_j}\right)\right| + \sum_{|z/z_j|<1/2} \log \left| E_\nu\left(\frac{z}{z_j}\right)\right| \\ &=: S_1 + S_2. \end{aligned}$$

Observe that S_1 is a finite sum by the standard uniqueness theorem of analytic functions.

To estimate S_2, where $|\frac{z}{z_j}| < 1/2$, recall the property (3) of Weierstraß products from Chapter 5. By this property,

$$\left| E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1},$$

hence

$$\left|E_\nu\left(\frac{z}{z_j}\right)\right| \le 1 + \left|\frac{z}{z_j}\right|^{\nu+1}.$$

Therefore,

$$\sum_{|z/z_j|<1/2} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right| \le \sum_{|z/z_j|<1/2} \log\left(1+\left|\frac{z}{z_j}\right|^{\nu+1}\right) \le \sum_{|z/z_j|<1/2} \left|\frac{z}{z_j}\right|^{\nu+1}. \quad (11.2)$$

We now have to analyze all cases in the definition of ν above. In the middle case, the sum (11.2) is majorized by

$$= \sum_{|z/z_j|<1/2} \left|\frac{z}{z_j}\right|^{\lambda} = |z|^\lambda \sum_{|z/z_j|<1/2} |z_j|^{-\lambda} = O(r^{\lambda+\varepsilon}),$$

since $\sum |z_j|^{-\lambda}$ converges. In the remaining two cases, $\nu + 1 > \lambda + \varepsilon$ for ε small enough and so

$$\left|\frac{z}{z_j}\right|^{\nu+1} = |z|^{\lambda+\varepsilon}\left|\frac{z}{z_j}\right|^{\nu+1-\lambda-\varepsilon} |z_j|^{-(\lambda+\varepsilon)} \le |z|^{\lambda+\varepsilon}|z_j|^{-(\lambda+\varepsilon)}.$$

Hence, the sum in (11.2) is now

$$\le |z|^{\lambda+\varepsilon} \sum_{|z/z_j|<1/2} |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}),$$

since $\sum |z_j|^{-(\lambda+\varepsilon)}$ converges by the definition of the exponent of convergence.

To estimate S_1, we first consider the case $\nu = 0$; recall that S_1 is a finite sum. Then

$$\begin{aligned} S_1 &= \sum_{|z/z_j|\ge 1/2} \log\left|E_0\left(\frac{z}{z_j}\right)\right| = \sum_{|z/z_j|\ge 1/2} \log\left|1-\frac{z}{z_j}\right| \\ &\le \sum_{|z/z_j|\ge 1/2} \log\left(1+\left|\frac{z}{z_j}\right|\right) \le A \sum_{|z/z_j|\ge 1/2} \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|^\varepsilon \sum_{|z/z_j|\ge 1/2} |z_j|^{-\varepsilon}, \quad (11.3) \end{aligned}$$

where A is a suitable constant. If $\lambda = 0$, then $\sum |z_j|^{-\varepsilon}$ converges and by (11.3),

$$S_1 = O(r^\varepsilon) = O(r^\lambda + \varepsilon).$$

If $\lambda = 1$ and $\sum |z_j|^{-1}$ converges, we get

$$\begin{aligned} S_1 &= A\sum\left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|\sum\left|\frac{z}{z_j}\right|^{\varepsilon-1}|z_j|^{-1} = A|z|\sum\left|\frac{z_j}{z}\right|^{1-\varepsilon}|z_j|^{-1} \\ &\le 2A|z|\sum|z_j|^{-1} = O(r^\lambda) = O(r^{\lambda+\varepsilon}), \end{aligned}$$

provided $\varepsilon < 1$. Since $\nu = 0$, we must have $\lambda \le 1$. Thus, assume now $\lambda \in (0,1)$ and take $\varepsilon < \lambda$. Then

$$S_1 = A \sum \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|^{\lambda+\varepsilon} \sum \left|\frac{z}{z_j}\right|^{-\lambda} |z_j|^{-(\lambda+\varepsilon)} \le A|z|^{\lambda+\varepsilon} \sum \left|\frac{z_j}{z}\right|^{\lambda} |z_j|^{-(\lambda+\varepsilon)}$$
$$\le 2A|z|^{\lambda+\varepsilon} \sum |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}).$$

Finally, we have to consider the case $\nu > 0$. Then, for each term in S_1,

$$\log \left| E_\nu\left(\frac{z}{z_j}\right)\right| \le \log\left|1 - \frac{z}{z_j}\right| + \left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^{\nu}$$
$$\le 2\left(\left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^{\nu}\right) \le 2\left|\frac{z}{z_j}\right|^{\nu}\left(1 + \left|\frac{z_j}{z}\right| + \cdots + \left|\frac{z_j}{z}\right|^{\nu-1}\right)$$
$$\le 2\left|\frac{z}{z_j}\right|^{\nu}(1 + 2 + \cdots + 2^{\nu-1}) \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\nu}.$$

If now $\nu = \lambda - 1$, then

$$\log \left| E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda-1} = 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda}\left|\frac{z_j}{z}\right| \le 2^{\nu+2}\left|\frac{z}{z_j}\right|^{\lambda}. \tag{11.4}$$

If $\nu \ne \lambda - 1$, and ε is small enough, then $\nu < \lambda + \varepsilon \le \nu + 1$ and $\lambda + \varepsilon + 1 \le \nu + 2$. Therefore,

$$\log \left| E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\nu} = 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon}\left|\frac{z_j}{z}\right|^{\lambda+\varepsilon-\nu}$$
$$\le 2^{\nu+1+\lambda+\varepsilon-\nu}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon} \le 2^{\nu+2}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon}. \tag{11.5}$$

From (11.4) and (11.5),

$$\sum_{|z/z_j|\ge 1/2} \log \left| E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+2} r^{\lambda+\varepsilon} \sum_{|z/z_j|\ge 1/2} |z_j|^{-(\lambda+\varepsilon)}$$
$$\le 2^{\nu+2} r^{\lambda+\varepsilon} \sum_{z_j} |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}).$$

So, we see that $S_1 = O(r^{\lambda+\varepsilon})$, $S_2 = O(r^{\lambda+\varepsilon})$. This means that

$$\log|Q(z)| = O(r^{\lambda+\varepsilon}),$$

hence

$$\log M(r,Q) = O(r^{\lambda+\varepsilon}),$$

and so

$$\rho(Q) = \limsup_{r\to\infty} \frac{\log\log M(r,Q)}{\log r} \le \lambda + \varepsilon. \quad \square$$

Theorem 11.2. (Hadamard). *Let $f(z)$ be a non-constant entire function of finite order ρ. Then*

$$f(z) = z^m Q(z) e^{P(z)},$$

where (1) $m \geq 0$ is the multiplicity of the zero of $f(z)$ at $z = 0$, (2) $Q(z)$ is the canonical product formed with the non-zero zeros of $f(z)$ and (3) $P(z)$ is a polynomial of degree $\leq \rho$.

Before we can prove the Hadamard theorem, we need the following

Lemma 11.3. *Let $Q(z)$ be a canonical product of order $\lambda = \lambda(Q)$. Given $\varepsilon > 0$, there exists a sequence $(r_n) \to +\infty$ such that for each r_n, the minimum modulus satisfies*

$$\mu(r_n) := \min_{|z|=r_n} |Q(z)| > e^{-r_n^{\lambda+\varepsilon}}. \tag{11.4}$$

Proof. Let (z_j) denote the zeros of $Q(z)$, $0 < |z_1| \leq |z_2| \leq \cdots$. Denote $r_j = |z_j|$. By the definition of the exponent of convergence, $\sum_j r_j^{-(\lambda+\varepsilon)}$ converges. This means that the length of the set

$$E := \bigcup_{j=1}^{\infty} \left[r_j - \frac{1}{r_j^{\lambda+\varepsilon}}, r_j + \frac{1}{r_j^{\lambda+\varepsilon}} \right]$$

is finite. We proceed to prove that (11.4) holds outside of E for all r sufficiently large. From the proof of Theorem 11.1,

$$\log |Q(z)| = S_1 + S_2' = \sum_{|z/z_j| \geq 1/2} \log \left| E_\nu\left(\frac{z}{z_j}\right) \right| + \log \prod_{|z/z_j|<1/2} \left| E_\nu\left(\frac{z}{z_j}\right) \right|.$$

Moreover, from the same proof, making use of the estimate for S_2, $S_2' \leq S_2 = O(r^{\lambda+\varepsilon})$. Recall now again that S_1 is a finite sum. Therefore,

$$S_1 = \sum_{|z/z_j| \geq 1/2} \log \left| 1 - \frac{z}{z_j} \right| + \sum_{|z/z_j| \geq 1/2} \log |e^{Q_\nu(z)}| =: S_{11} + S_{12}.$$

Assume now that $r \notin E$ is sufficiently large. Then, as $2r \geq r_j$

$$\left| 1 - \frac{z}{z_j} \right| = \frac{|z_j - z|}{|z_j|} \geq \frac{|r - r_j|}{r_j} \geq r_j^{-1-\lambda-\varepsilon} \geq (2r)^{-1-\lambda-\varepsilon}$$

and so

$$S_{11} = \sum_{|z/z_j| \geq 1/2} \log \left| 1 - \frac{z}{z_j} \right| \geq -(1 + \lambda + \varepsilon)\big(\log(2r)\big) n(2r).$$

By Theorem 9.8, $n(2r) = O(r^{\lambda+\varepsilon})$. Since $r^\varepsilon > (1 + \lambda + \varepsilon) \log 2r$ for r sufficiently large, we get

$$S_{11} \geq -r^{\lambda+3\varepsilon}.$$

For S_{12}, we may apply the proof of Theorem 11.1 to see that

$$S_{12} < S_1 = O(r^{\lambda+\varepsilon}).$$

Writing this as $S_{12} \le K r^{\lambda+\varepsilon}$ for r large enough, we get

$$\log|Q(z)| \ge -r^{\lambda+3\varepsilon} - K(r^{\lambda+\varepsilon}) = -r^{\lambda+3\varepsilon}(Kr^{-2\varepsilon}+1) \ge -2r^{\lambda+3\varepsilon} \ge -r^{\lambda+4\varepsilon}.$$

By exponentiation, we get

$$|Q(z)| \ge e^{-r^{\lambda+4\varepsilon}},$$

hence (11.4) holds. □

Proof of Theorem 11.2. By the construction of the canonical product, $z^m Q(z)$ has exactly the same zeros as $f(z)$, with the same multiplicities as well. Therefore,

$$f(z)/z^m Q(z)$$

is an entire function with no zeros. By Theorem 4.1, there is an entire function $g(z)$ such that

$$f(z) = z^m Q(z) e^{g(z)}.$$

It remains to prove that $g(z)$ is a polynomial of degree $\le \rho$. Since $f(z)$ is of order ρ,

$$M(r,f) \le e^{r^{\rho+\varepsilon}}$$

for all r sufficiently large. Now the order of $Q(z) = \lambda = \lambda(f) \le \rho$. Take r such that (11.4) is true. Then

$$\max_{|z|=r} |e^{g(z)}| = \max_{|z|=r} e^{\operatorname{Re} g(z)} \le \frac{\max_{|z|=r}|f(z)|}{r^m \min_{|z|=r}|Q(z)|} \le \frac{e^{r^{\rho+\varepsilon}}}{e^{-r^{\lambda+\varepsilon}}} = e^{r^{\rho+\varepsilon}} \cdot e^{r^{\lambda+\varepsilon}} \le e^{2r^{\rho+\varepsilon}}.$$

Recalling Definition 7.4, we observe that

$$A(r,g) \le 2r^{\rho+\varepsilon}.$$

By Theorem 7.6, g is a polynomial of degree $\le \rho+\varepsilon$, hence $\le \rho$. □

Corollary 11.4. *Let $f(z)$ be a nonconstant entire function of finite order ρ which is no natural number. Then $\lambda(f) = \rho$.*

Proof. If $\rho = 0$, then by Theorem 11.2, $\deg P(z) = 0$, hence $P(z)$ is a constant. Therefore,

$$\rho = \rho(Q) = \lambda(Q) = \lambda(f).$$

Assume now that $\lambda(f) < \rho$. By Theorem 11.2, $\deg P(z) \le \rho \notin \mathbb{N}$, hence $\deg P(z) < \rho$. By Lemma 7.2,

$$M(r, e^P) \le e^{2|a_n| r^n};$$

here now $P(z) = a_n z^n + \cdots + a_0$. Therefore $\rho(e^P) \le n$. On the other hand,

$$M(r, e^P) = \max_{|z|=r} |e^P| = e^{\max_{|z|=r} \operatorname{Re} P} = e^{A(r,P)} \ge e^{Kr^n}$$

for some $K > 0$ by Theorem 7.5. Hence $\rho(e^P) \ge n$, and so $\rho(e^P) = n < \rho$. By Theorem 7.9,

$$\rho(f) \le \max\bigl(\rho(z^m), \rho(Q), \rho(e^P)\bigr) \le \max\bigl(\lambda(f), n\bigr) < \rho = \rho(f),$$

a contradiction. □

Corollary 11.5. *If $f(z)$ is transcendental entire and $\rho(f) \notin \mathbb{N}$, then $f(z)$ has infinitely many zeros.*

Proof. If $\rho > 0$, then $\lambda(f) > 0$, and so f must have infinitely many zeros. If then $\rho = 0$, the Hadamard theorem implies that $f(z) = cz^m Q(z)$, $c \in \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$. Since $f(z)$ is not a polynomial, $Q(z)$ cannot be a polynomial and $\rho(Q) = 0$. By the construction of a canonical product, $Q(z)$ is the product of terms of type $E_0(\frac{z}{z_j})$. Since it is not a polynomial, the number of zeros z_j must be infinite. □