2. The residue theorem

Let $z = a$ be an isolated singularity of f and let

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j$$

be its Laurent expansion around $z = a$. Define now the *residue of f at $z = a$* by

$$\operatorname{Res}(f, a) := a_{-1}.$$

Theorem 2.1. (Residue theorem). *Assume that $f\colon G \to \overline{\mathbb{C}}$ is analytic in a convex region G except for finitely many poles $a_1, \dots, a_n$ and let γ be a piecewise continuously differentiable closed path in G such that $a_j \notin \gamma(I)$, $j = 1, \dots, n$. Then*

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta = \sum_{j=1}^{n} n(\gamma, a_j) \operatorname{Res}(f, a_j),$$

where $n(\gamma, a_j)$ denotes the winding number of γ around $z = a_j$ counterclockwise.

Remark. 1) Intuitively, the winding number tells how many times one goes around $z = a_j$ as one follows the path γ from $\gamma(0)$ to $\gamma(1)$. We omit the exact definition.

(2) The residue theorem holds good even in a number of more general situations. We omit these considerations.

Proof of Theorem 2.1. Let

$$f(z) = \sum_{j=-\mu_k}^{\infty} a_{j,k}(z-a_k)^j = S_k(z) + \sum_{j=0}^{\infty} a_{j,k}(z-a_k)^j$$

be the Laurent expansions of $f(z)$ around $z = a_k$, $k = 1, \dots, n$. Clearly, $g(z) = f(z) - \sum_{k=1}^{n} S_k(z)$ is analytic in G. By the Cauchy theorem,

$$\begin{aligned} 0 = \int_\gamma g(\zeta)\, d\zeta &= \int_\gamma f(\zeta)\, d\zeta - \sum_{k=1}^{n} \int_\gamma S_k(\zeta)\, d\zeta \\ &= \int_\gamma f(\zeta)\, d\zeta - \sum_{k=1}^{n} \sum_{j=-\mu_k}^{-1} a_{j,k} \int_\gamma (\zeta - a_k)^j\, d\zeta. \end{aligned}$$

Therefore, it suffices to compute

$$\int_\gamma (\zeta - a_k)^{-m}\, d\zeta$$

for $1 \le k \le n$ and for any $m \in \mathbb{N}$. This integral is independent of the path and so we may assume γ to be a circle centered at a_k. Since $(\zeta - a_k)^{-m}$ has a primitive for $m \ge 2$, then $\int_\gamma (\zeta - a_k)^{-m} = 0$ for $m \ge 2$. If $m = 1$, then

$$\int_\gamma (\zeta - a_k)^{-1}\, d\zeta = 2\pi i n(\gamma, a_k)$$

by the Cauchy integral formula. Therefore,

$$\begin{aligned} 0 &= \int_\gamma f(\zeta)\, d\zeta - \sum_{k=1}^{n} a_{-1,k} \cdot 2\pi i n(\gamma, a_k) \\ &= \int_\gamma f(\zeta)\, d\zeta - 2\pi i \sum_{k=1}^{n} n(\gamma, a_j) \operatorname{Res}(f, a_j). \qquad \square \end{aligned}$$

Theorem 2.2. *If $f(z)$ has a pole of multiplicity m at $z = a$ and $g(z) := (z - a)^m f(z)$, then*

$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Proof. Clearly,

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-a)^j$$

and so

$$g(z) = a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + \cdots,$$

hence

$$g^{(m-1)}(a) = (m-1)!\, a_{-1}. \quad \square$$

Corollary 2.3. *If $f(z)$ has a simple pole at $z = a$ and $g(z) := (z-a)f(z)$, then*

$$\operatorname{Res}(f, a) = g(a) = \lim_{z \to a} (z-a) f(z).$$

Example 2.4. To compute

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2},$$

consider

$$f(z) = \frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} + \frac{1}{z+i}\right).$$

$f(z)$ is analytic in $\mathbb{C} \setminus \{i, -i\}$, with simple poles at $z = \pm i$. By Corollary 2.3,

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z-i) f(z) = \frac{1}{2i}.$$

Assume $R > 1$, and compute $\int_\gamma f(\zeta)\, d\zeta$, where γ is as in the figure. By the residue theorem

$$\int_\gamma \frac{d\zeta}{1+\zeta^2} = 2\pi i \operatorname{Res}(f, i) = \pi.$$

On the other hand,

$$\int_\gamma \frac{d\zeta}{1+\zeta^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{K_R} \frac{d\zeta}{1+\zeta^2},$$

where K_R is the half-circle part of γ. But $\zeta = Re^{i\varphi}$ on γ and so $d\zeta = iRe^{i\varphi}\, d\varphi$, hence

$$\left| \int_{K_R} \frac{d\zeta}{1+\zeta^2} \right| = \left| \int_0^\pi \frac{iRe^{i\varphi}}{1+\zeta}\, d\varphi \right| \le R \int_0^\pi \frac{d\varphi}{|1+\zeta^2|} \le \frac{R\pi}{R^2-1} \to 0 \qquad \text{as } R \to \infty,$$

since $|1+\zeta^2| \ge \big| |\zeta|^2 - 1 \big| = R^2 - 1$ on K_R. Therefore

$$\pi = \lim_{R\to\infty} \int_\gamma \frac{d\zeta}{1+\zeta^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + \lim_{R\to\infty} \int_{K_R} \frac{d\zeta}{1+\zeta^2},$$

giving

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi. \quad \square$$

Example 2.5. Prove that

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

Now

$$f(z) = \frac{z^2}{1+z^4}$$

is analytic in $\mathbb{C} \setminus \{a_1, \ldots, a_4\}$, where a_j:s are the fourth roots of -1. Making use of the same path γ as in Example 2.4, we need a_1, a_2 only;

$$a_1 = \frac{1}{\sqrt{2}}(1+i), \qquad a_2 = \frac{1}{\sqrt{2}}(1-i).$$

Now,

$$\begin{aligned}\operatorname{Res}(f,a_1) &= \lim_{z\to a_1}(z-a_1)f(z) = \lim_{z\to a_1}(z-a_1)\frac{z^2}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)} \\ &= \frac{a_1^2}{(a_1-a_2)(a_1-a_3)(a_1-a_4)} = \frac{1-i}{4\sqrt{2}}.\end{aligned}$$

Similarly,

$$\operatorname{Res}(f,a_2) = \frac{-1-i}{4\sqrt{2}}.$$

By the residue theorem,

$$\frac{1}{2\pi i}\int_\gamma f(\zeta)\,d\zeta = \operatorname{Res}(f,a_1) + \operatorname{Res}(f,a_2) = -\frac{i}{2\sqrt{2}}.$$

On the other hand,

$$\frac{1}{2\pi i}\int_\gamma f(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_{-R}^{R}\frac{x^2\,dx}{1+x^4} + \frac{1}{2\pi i}\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4}.$$

Now,

$$\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4} = \int_0^\pi \frac{R^2e^{2i\varphi}}{1+R^4e^{4i\varphi}}\cdot Rie^{i\varphi}\,d\varphi = \int_0^\pi iR^3\frac{e^{3i\varphi}\,d\varphi}{1+R^4e^{4i\varphi}}.$$

Since $|1+R^4e^{4i\varphi}| \geq R^4-1$, we get

$$\left|\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4}\right| \leq \frac{R^3}{R^4-1}\int_0^\pi d\varphi = \frac{\pi R^3}{R^4-1} \to 0 \qquad \text{as } R\to\infty$$

and so

$$-\frac{i}{2\sqrt{2}} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{x^2\,dx}{1+x^4} \implies \int_{-\infty}^{\infty}\frac{x^2\,dx}{1+x^4} = \frac{\pi}{\sqrt{2}}. \quad \square$$

Example 2.6. Compute

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi} \qquad \text{for } a>1.$$

On the unit circle $|z|=1$, $z=e^{i\varphi}$ and so $\overline{z}=e^{-i\varphi}=\frac{1}{e^{i\varphi}}=\frac{1}{z}$ and

$$\frac{z^2+2az+1}{2z}=a+\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)=a+\tfrac{1}{2}(z+\overline{z})=a+\tfrac{1}{2}(e^{i\varphi}+e^{-i\varphi})=a+\cos\varphi.$$

Let γ be the unit circle. Observing that $\cos(-\varphi)=\cos\varphi$, we get

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi}=\tfrac{1}{2}\int_0^{2\pi}\frac{d\varphi}{a+\cos\varphi}=-i\int_\gamma \frac{dz}{z^2+2az+1} \tag{2.1}$$

Now, $z^2+2az+1=(z-\alpha)(z-\beta)$, where

$$\alpha=-a+\sqrt{a^2-1}, \qquad \beta=-a-\sqrt{a^2-1}.$$

Since $a>1$, it is easy to see that $|\alpha|<1$, $|\beta|>1$. Therefore, by the residue theorem,

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+2az+1}&=2\pi i\,\mathrm{Res}(f,\alpha)=2\pi i\lim_{z\to a}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\\&=2\pi i\frac{1}{\alpha-\beta}=\frac{\pi i}{\sqrt{a^2-1}}.\end{aligned}$$

Combining with (2.1), one obtains

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi}=\frac{\pi}{\sqrt{a^2-1}}. \quad \square$$

Example 2.7. To evaluate

$$\int_0^\infty \frac{\sin x}{x}\,dx,$$

we consider

$$\begin{aligned}\int_\gamma \frac{e^{iz}}{z}\,dz&=\int_{-R}^{-\rho}\frac{e^{ix}}{x}\,dx+\int_{-\gamma_1}\frac{e^{iz}}{z}\,dz+\int_\rho^R\frac{e^{ix}}{x}\,dx+\int_{\gamma_2}\frac{e^{iz}}{z}\,dz\\&=2i\int_\rho^R\frac{\sin x}{x}\,dx+\int_{\gamma_1}\frac{e^{iz}}{z}\,dz+\int_{\gamma_2}\frac{e^{iz}}{z}\,dz.\end{aligned}$$

The integral $=0$, since (1) $f(z)=e^{iz}/z$ is analytic inside of γ, (2) $e^{iz}=\cos z+i\sin z$, (3) $\cos x/x$ is an odd function and $\sin x/x$ is even.

To evaluate the integral over γ_2, we need the Jordan inequality

$$\int_0^\pi e^{-R\sin\varphi}\,d\varphi\le\frac{\pi}{R}(1-e^{-R}) \qquad (R>0).$$

To this end, consider $g(\varphi) := \sin\varphi - \varphi\cos\varphi$. Since $g(0) = 0$ and $g'(\varphi) = \cos\varphi - \cos\varphi + \varphi\sin\varphi \geq 0$, $g(\varphi) \geq 0$ for $0 \leq \varphi \leq \pi/2$. Therefore,

$$D\left(\frac{\sin\varphi}{\varphi}\right) = \frac{\varphi\cos\varphi - \sin\varphi}{\varphi^2} \leq 0, \qquad 0 < \varphi \leq \pi/2;$$

since $(\sin\varphi/\varphi)_{\varphi=\pi/2} = \frac{2}{\pi}$, we have $\sin\varphi/\varphi \geq \frac{2}{\pi}$ for $0 < \varphi \leq \pi/2$. Then $e^{-R\sin\varphi} \leq e^{-R\frac{2\varphi}{\pi}}$, and so

$$\int_0^\pi e^{-R\sin\varphi}\,d\varphi = 2\int_0^{\pi/2} e^{-R\sin\varphi}\,d\varphi \leq 2\int_0^{\pi/2} e^{-R\cdot\frac{2\varphi}{\pi}}\,d\varphi = \frac{\pi}{R}(1 - e^{-R}).$$

Therefore,

$$\left|\int_{\gamma_2} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^\pi e^{iR(\cos\varphi + i\sin\varphi)}\cdot i\,d\varphi\right| \leq \int_0^\pi |e^{iR\cos\varphi}|e^{-R\sin\varphi}\,d\varphi$$
$$= \int_0^\pi e^{-R\sin\varphi}\,d\varphi \leq \frac{\pi}{R}(1 - e^{-R}) \to 0 \qquad \text{as } R \to \infty.$$

By the Taylor expansion of e^{iz},

$$\frac{e^{iz}}{z} = \frac{1}{z} + g(z), \qquad g(z) \text{ analytic (in } \mathbb{C}).$$

So,

$$\int_{\gamma_1} \frac{e^{iz}}{z}\,dz = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_1} g(z)\,dz,$$

and now

$$\int_{\gamma_1} \frac{dz}{z} = i\int_0^\pi d\varphi = \pi i,$$
$$\left|\int_{\gamma_1} g(z)\,dz\right| \leq K\int_0^\pi |\rho e^{i\varphi}|\,d\varphi = K\pi\rho \to 0 \qquad \text{as } \rho \to 0.$$

Therefore,

$$\int_{\gamma_1} \frac{e^{iz}}{z}\,dz \to \pi i \qquad \text{as } \rho \to 0.$$

Hence,

$$0 = 2i\int_\rho^R \frac{\sin x}{x}\,dx - \int_{\gamma_1} \frac{e^{iz}}{z}\,dz + \int_{\gamma_2} \frac{e^{iz}}{z}\,dz$$
$$\to 2i\int_0^\infty \frac{\sin x}{x}\,dx - \pi i \qquad \text{as } R \to \infty \text{ and } \rho \to 0.$$

This results in

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \quad \square$$

Example 2.8. Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

Consider

$$f(z) = \frac{1+2iz-e^{2iz}}{z^2}.$$

The only possible pole is $z=0$. Since the power series of e^{2iz} converges for all z (e^{2iz} is entire!), $\varphi(z)$ below is bounded around $z=0$:

$$\begin{aligned}\frac{1+2iz-e^{2iz}}{z^2} &= \frac{1}{z^2}+\frac{2i}{z}-\left(\frac{e^{iz}}{z}\right)^2 = \frac{1}{z^2}+\frac{2i}{z}-\left(\frac{1}{z}+i-\tfrac{1}{2}z+\cdots\right)^2 \\ &= \frac{1}{z^2}+\frac{2i}{z}-\frac{1}{z^2}-\frac{2i}{z}+\varphi(z);\end{aligned}$$

Therefore, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\varphi(z) = 0$, and so $f(z)$ has a removable singularity at $z=0$. Since $f(z)$ is analytic in $\mathbb{C}$, by the Cauchy theorem,

$$0 = \int_\gamma f(\zeta)\,d\zeta = \int_{\frown{\gamma}} f(\zeta)d\zeta + \int_{-R}^{R} \frac{1+2ix-e^{2ix}}{x^2}\,dx.$$

For the integral on $\vec{\gamma}$, we get

$$\begin{aligned}\int_{-R}^{R}\frac{1+2ix-e^{2ix}}{x^2}\,dx &= \int_{-R}^{R}\frac{1-e^{2ix}}{x^2}\,dx + 2i\int_{-R}^{R}\frac{dx}{x} \\ &= \int_{-R}^{R}\frac{1-\cos 2x}{x^2}\,dx - i\int_{-R}^{R}\frac{\sin 2x}{x^2}\,dx + 2i\int_{-R}^{R}\frac{dx}{x} \\ &= 2\int_{-R}^{R}\frac{\sin^2 x}{x^2}\,dx + \text{ a purely imaginary term} \\ &= 4\int_0^R \frac{\sin^2 x}{x^2}\,dx + \text{ a purely imaginary term.}\end{aligned}$$

For the integral on $\frown{\gamma}$,

$$\begin{aligned}\int_{\frown{\gamma}} f(\zeta)\,d\zeta &= \int_0^\pi \frac{1+2iRe^{i\varphi}-e^{2iRe^{i\varphi}}}{R^2e^{2i\varphi}}\cdot iRe^{i\varphi}\,d\varphi \\ &= \int_0^\pi \frac{i}{R}e^{-i\varphi}\,d\varphi - 2\int_0^\pi d\varphi - \int_0^\pi \frac{i}{R}e^{-i\varphi}e^{2iRe^{i\varphi}}\,d\varphi = I_1+I_2+I_3.\end{aligned}$$

Now,

$$\begin{aligned}|I_1| &\le \frac{1}{R}\int_0^\pi d\varphi = \frac{\pi}{R} \to 0 \qquad \text{as } R\to\infty, \\ I_2 &= -2\pi\end{aligned}$$

and

$$\begin{aligned}|I_3| &= \left|\int_0^\pi \frac{i}{R} e^{-i\varphi} e^{2iR\cos\varphi} e^{-2R\sin\varphi}\, d\varphi\right| \\ &\leq \frac{1}{R}\int_0^\pi e^{-2R\sin\varphi}\, d\varphi = \frac{2}{R}\int_0^{\pi/2} e^{-2R\sin\varphi}\, d\varphi \\ &\leq \frac{2}{R}\int_0^{\pi/2} e^{-\frac{4R\varphi}{\pi}}\, d\varphi = \frac{\pi}{2R^2}(1-e^{-2R}) \to 0 \qquad \text{as } R\to\infty.\end{aligned}$$

Therefore, by taking real parts,

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \lim_{R\to\infty}\left(-\tfrac{1}{4}\int_{\widehat{\gamma}} f(\zeta)\, d\zeta\right) = \frac{\pi}{2} + \lim_{R\to\infty}(I_1+I_3) = \frac{\pi}{2}. \quad \square$$

Example 2.9. To compute,

$$\int_0^\infty \frac{dx}{(x^2+1)^2},$$

denote

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}.$$

Clearly, $f(z)$ has double poles in $z=\pm i$, and no other poles. Therefore, by Theorem 2.2,

$$\operatorname{Res}(f,i) = \frac{1}{1!}g'(i),$$

where $g(z) = (z-i)^2 f(z) = \frac{1}{(z+i)^2}$. Hence,

$$\big(g'(z)\big)_{z=i} = \left(-\frac{2}{(z+i)^3}\right)_{z=i} = \frac{1}{4i}$$

and so

$$\operatorname{Res}(f,i) = \frac{1}{4i}.$$

By the residue theorem,

$$\int_\gamma \frac{d\zeta}{(\zeta^2+1)^2} = 2\pi i \operatorname{Res}(f,i) = \frac{\pi}{2}.$$

On the other hand,

$$\int_\gamma \frac{d\zeta}{(\zeta^2+1)^2} = \int_{-R}^R \frac{dx}{(x^2+1)^2} + \int_{K_R} \frac{d\zeta}{(\zeta^2+1)^2}.$$

But

$$\left|\int_{K_R} \frac{d\zeta}{(\zeta^2+1)^2}\right| \leq \frac{\pi R}{(R^2-1)^2} \to 0 \qquad \text{as } R\to\infty.$$

Since $\frac{1}{(x^2+1)^2}$ is an even function,

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \tfrac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = \tfrac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Exercises. Evaluate the following integrals by making use of the residue theorem

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^3}$,

(2) $\displaystyle\int_{0}^{\pi/2} \frac{d\varphi}{a+\sin^2\varphi}$ for $a>0$,

(3) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^3}\,dx$,

(4) $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$.

Additional reading:

D. Mitrinović: Calculus of Residues, Groningen 1966.