3. The argument principle

3.1. The logarithm in the complex plane. The exponential function is locally injective in $\mathbb{C}$. In fact, assume

$$e^z = e^t \implies e^{z-t} = 1.$$

Denote $z - t = x + iy$, $x, y \in \mathbb{R}$. Then

$$1 = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = 1$$

$\implies$

$$\begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0. \end{cases}$$

Since

$$1 = |e^{z-t}| = e^x,$$

we see that $x = 0$. Then $\cos y = 1$, $\sin y = 0$ implies $y = n \cdot 2\pi$. Therefore, the nearest possible points z, t with $e^z = e^t$ have a distance 2π, and given any z_0, e^z is injective in $B(z_0, 2\pi)$.

So, we can locally define the inverse function $\log z$ for the exponential. Since

$$z = e^{\log z} = e^{\log z + n \cdot 2\pi i},$$

$\log z$ has infinitely many branches. Denoting $u + iv = \log z$, we get

$$z = e^{u+iv} = e^u e^{iv} \implies |z| = e^u \implies u = \log |z|$$

and

$$re^{i\varphi} = z = |z|e^{i\varphi} = e^u e^{iv}$$

and so we may take $v = \varphi = \arg z$. Hence

$$\log z = \log |z| + i \arg z + n \cdot 2\pi i$$

If γ is now a closed path in $\mathbb{C}$, and we consider $\log z$ on γ, we easily see that return to the original branch appears, if the winding number around $z = 0$ is zero; otherwise we move to another branch. So, if we have a domain $G \subset \mathbb{C} \setminus \{0\}$, then $\log z$ is uniquely determined and analytic in G. This will be applied in the proof of Theorem 3.3.

3.2. The argument principle. Assume $f(z)$ is analytic around $z = a$ and has a zero of multiplicity m at $z = a$. Then $f(z) = (z - a)^m g(z)$, $g(a) \neq 0$. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{3.1}$$

Since $g(a) \neq 0$, $g'(z)/g(z)$ is analytic around $z = a$. Similarly, if $f(z)$ has a pole of order m at $z = a$, and $f(z) = (z - a)^{-m} g(z)$, $g(a) \neq 0$, then

$$\frac{f'(z)}{f(z)} = -\frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{3.2}$$

Definition 3.1. Assume that $f\colon G \to \overline{\mathbb{C}}$ is analytic in an open set $G \subset \mathbb{C}$ except for poles. Then f is said to be *meromorphic* in G.

Theorem 3.2. *Assume that $f\colon G \to \overline{\mathbb{C}}$ is meromorphic in a convex region G except for finitely many zeros $a_1, \dots, a_n$ and poles $b_1, \dots, b_n$, each counted according to their multiplicity. If γ is a piecewise continuously differentiable closed path in G such that $a_j \notin \gamma(I)$, $j = 1, \dots, n$, and $b_j \notin \gamma(I)$, $j = 1, \dots, m$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{j=1}^{n} n(\gamma, a_j) - \sum_{j=1}^{m} n(\gamma, b_j).$$

Proof. By the same idea as in (3.1) and (3.2),

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z-a_j} - \sum_{j=1}^{m} \frac{1}{z-b_j} + \frac{g'(z)}{g(z)},$$

where $g(z)$ is analytic non-zero in G. Since g'/g is analytic in G, residue theorem and the Cauchy theorem result in

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_\gamma \frac{1}{\zeta-a_j}\,d\zeta - \frac{1}{2\pi i}\sum_{j=1}^{m}\int_\gamma \frac{1}{\zeta-b_j}\,d\zeta \\ &= \sum_{j=1}^{n} n(\gamma, a_j) - \sum_{j=1}^{m} n(\gamma, b_j). \quad \square\end{aligned}$$

Theorem 3.3. (Rouché). *Let f, g be meromorphic in a convex region G and let $\overline{B(a,R)} \subset G$ be a closed disc. Suppose f, g have no zeros and no poles on the circle $\gamma = \partial B(a,R) = \{\, z \in G \mid |z-a| = R \,\}$ and that $|f(z) - g(z)| < |g(z)|$ for all $z \in \gamma$. Then*

$$\mu_f - \nu_f = \mu_g - \nu_g,$$

where μ_f, μ_g, resp. ν_f, ν_g, are the number zeros, resp. poles, of f and g in $\{\, z \in G \mid |z-a| < R \,\}$, counted according to multiplicity.

Proof. By the assumption,

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \tag{3.3}$$

for all $z \in \gamma$. By the Theorem 3.2,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\big(f(\zeta)/g(\zeta)\big)'}{\big(f(\zeta)/g(\zeta)\big)}\,d\zeta &= \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}\,d\zeta \\ &= \mu_f - \nu_f - (\mu_f - \nu_g),\end{aligned}$$

since the winding number of γ for all zeros and poles in $\{\, z \in G \mid |z-a| < R \,\}$ equals to one. On the other hand, by (3.3), f/g maps γ into $B(1,1)$, and so a fixed branch of $\log(f/g)$ is a primitive of $(f/g)'/(f/g)$. Integrating over γ, the logarithm doesn't change the branch, hence $\log(f/g)$ takes the same value at $\gamma(0)$ and $\gamma(1) = \gamma(0)$ resulting in

$$\frac{1}{2\pi i}\int_\gamma \frac{\big(f(\zeta)/g(\zeta)\big)'}{\big(f(\zeta)/g(\zeta)\big)}\,d\zeta = 0.$$

The assertion now follows. $\square$