4. Infinite products

The basic idea here is to separate the zeros (and poles) of a meromorphic function $f(z)$ as a product component of $f(z)$. In principle, this results in an infinite product. To this end, we first prove

Theorem 4.1. *If $f(z)$ is an entire function with no zeros, then there exists another entire function $g(z)$ such that*

$$f(z) = e^{g(z)}.$$

Proof. Since $f(z) \neq 0$ for all $z \in \mathbb{C}$, then $\frac{f'(z)}{f(z)}$ is entire. Therefore,

$$\frac{f'(z)}{f(z)} = \sum_{j=0}^{\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + \cdots .$$

is a power series representation converging in the whole $\mathbb{C}$. Consider

$$h(z) = a_0 z + \tfrac{1}{2} a_1 z^2 + \tfrac{1}{3} a_3 z^3 + \cdots = z(a_0 + \tfrac{1}{2} a_1 z + \tfrac{1}{3} a_3 z^2 + \cdots). \tag{4.1}$$

Since

$$\limsup_{j\to\infty} \sqrt[j]{\frac{1}{j+1}|a_j|} = \limsup_{j\to\infty} \frac{1}{\sqrt[j]{j+1}} \sqrt[j]{|a_j|} = \limsup_{j\to\infty} \sqrt[j]{|a_j|} = 0,$$

hence the power series (4.1) has radius of convergence $= \infty$. Therefore, (4.1) determines an entire function. Differentiating term by term, as we may do for a converging power series, we get

$$h'(z) = \frac{f'(z)}{f(z)}.$$

Define now

$$\varphi(z) := f(z) e^{-h(z)},$$

hence

$$\varphi'(z) = f'(z) e^{-h(z)} - f(z) h'(z) e^{-h(z)} = e^{-h(z)} \big(f'(z) - f(z) h'(z) \big) \equiv 0.$$

Therefore, $\varphi(z)$ is constant, say $\varphi(z) \equiv e^a$, $a \in \mathbb{C}$. Note that $\varphi(z) \neq 0$ for all $z \in \mathbb{C}$. So,

$$f(z) e^{-h(z)} = e^a \implies f(z) = e^{a+h(z)}.$$

Defining $g(z) := a + h(z)$, we have the assertion. □

Definition 4.2. The infinite product $\prod_{j=1}^{\infty} b_j$ of complex numbers b_j *converges*, if there exists

$$\lim_{n\to\infty} \prod_{j=1}^{n} b_j \neq 0.$$

Remark. Define $P_n := \prod_{j=1}^n b_i$. Clearly, $\prod_{j=1}^\infty b_i$ converges if and only if (P_n) converges and $\lim_{n\to\infty} P_n \neq 0$. Then $b_n = P_n/P_{n-1}$ and there exists

$$\lim_{n\to\infty} b_n = \frac{\lim_{n\to\infty} P_n}{\lim_{n\to\infty} P_{n-1}} = 1. \tag{4.2}$$

Therefore, it is customary to use the notation

$$b_n = 1 + a_n;$$

then $\lim_{n\to\infty} a_n = 0$ by (4.2).

Theorem 4.3. *If $a_j \geq 0$ for all $j \in \mathbb{N}$, then $\prod_{j=1}^\infty (1 + a_j)$ converges if and only if $\sum_{j=1}^\infty a_j$ converges.*

Proof. Observe first that $P_n := \prod_{j=1}^n (1 + a_j)$ is a non-decreasing sequence, since $a_j \geq 0$. Therefore, (P_n) either converges to a finite (real) value, or to $+\infty$. Clearly,

$$a_1 + a_2 + \cdots + a_n \leq (1 + a_1)(1 + a_2) \cdots (1 + a_n).$$

On the other hand,

$$(1 + a_1) \cdots (1 + a_n) \leq e^{a_1} \cdots e^{a_n} = e^{a_1 + \cdots + a_n},$$

since $e^x \geq 1 + x$ for every $x \geq 0$. So, we have

$$\sum_{j=1}^n a_j \leq \prod_{j=1}^n (1 + a_j) \leq e^{\sum_{j=1}^n a_j}. \tag{4.3}$$

If $(\sum_{j=1}^n a_j)_{n\in\mathbb{N}}$ converges, then $(e^{\sum_{j=1}^n a_j})_{n\in\mathbb{N}}$ converges by the continuity of the exponential function. This implies that the increasing sequence $\big(\prod_{j=1}^n (1 + a_j)\big)_{n\in\mathbb{N}}$ converges to a non-zero limit by (4.3). If $\big(\prod_{j=1}^n (1 + a_j)\big)_{n\in\mathbb{N}}$ converges, then the increasing sequence $(\sum_{j=1}^n a_j)_{n\in\mathbb{N}}$ converges, again by (4.3). □

Theorem 4.4. *If $a_j \geq 0$, $a_j \neq 1$, for all $j \in \mathbb{N}$, then $\prod_{j=1}^\infty (1 - a_j)$ converges if and only if $\sum_{j=1}^\infty a_j$ converges.*

Proof. (1) Assume $\sum_{j=1}^\infty a_j$ converges. By the Cauchy criterium,

$$\sum_{j=N}^\infty a_j < \tfrac{1}{2}$$

for N sufficiently large; then also $a_j < 1$, $j \geq N$. Observe that

$$\begin{aligned}(1 - a_N)(1 - a_{N+1}) &= 1 - a_N - a_{N+1} + a_N a_{N+1} \\ &\geq 1 - a_N - a_{N+1} \quad \big(= 1 - (a_N + a_{N+1}) > \tfrac{1}{2}\big).\end{aligned}$$

Assume we have proved

$$(1 - a_N)(1 - a_{N+1}) \cdots (1 - a_n) \geq 1 - a_N - a_{N+1} - \cdots - a_n. \tag{4.4}$$

Then

$$
\begin{aligned}
(1-a_N)&(1-a_{N+1})\cdots(1-a_n)(1-a_{n+1})\\
&\geq (1-a_N-a_{N+1}-\cdots-a_n)(1-a_{n+1})\\
&= 1-a_N-a_{N+1}-\cdots-a_n-a_{n+1}+(a_N+\cdots+a_n)a_{n+1}\\
&\geq 1-a_N-a_{N+1}-\cdots-a_{n+1},
\end{aligned}
$$

and so (4.4) is true for all $n \geq N$. Therefore

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n) \geq 1-(a_N+\cdots+a_n) > \tfrac{1}{2}.$$

This implies that the decreasing sequence $\prod_{j=N}^{\infty}(1-a_j)$ converges to a limit $P \geq \frac{1}{2}$. If N is sufficiently large, then $0 < 1-a_j < 1$ and so $P \leq 1$. Writing, for $n > N$,

$$P_n = \prod_{j=1}^{n}(1-a_j) = P_{N-1}\cdot\prod_{j=N}^{n}(1-a_j),$$

we get

$$\lim_{n\to\infty} P_n = P_{N-1}\cdot\lim_{n\to\infty}\prod_{j=N}^{n}(1-a_j) = P_{N-1}\cdot P = (1-a_1)\cdots(1-a_{N-1})P \neq 0,$$

so $\prod_{j=1}^{\infty}(1-a_j)$ converges.

(2) Assume now that $\sum_{j=1}^{\infty} a_j$ diverges. If a_j does not converge to zero, then $1-a_j$ does not converge to one. By the Remark after Definition 4.2, $\prod_{j=1}^{\infty}(1-a_j)$ diverges.

So, we may assume that $\lim_{j\to\infty} a_j = 0$. Let N be sufficiently large so that $0 \leq a_j < 1$ for $j \geq N$. Since $1-x \leq e^{-x}$ for $0 \leq x < 1$, we have

$$1-a_j \leq e^{-a_j}, \qquad j \geq N.$$

Therefore,

$$0 \leq \prod_{j=N}^{n}(1-a_j) \leq \prod_{j=N}^{n} e^{-a_j} = e^{-\sum_{j=N}^{n} a_j}, \qquad n > N.$$

Since $\sum_{j=N}^{\infty} a_j$ diverges, $\lim_{n\to\infty}\sum_{j=N}^{n} a_j = +\infty$, and so $\lim_{n\to\infty} e^{-\sum_{j=N}^{n} a_j} = 0$, implying that

$$\lim_{n\to\infty}\prod_{j=1}^{n}(1-a_j) = 0.$$

By Definition 4.2, $\prod_{j=1}^{\infty}(1-a_j)$ diverges. $\square$

Definition 4.5. The infinite product $\prod_{j=1}^{\infty}(1+a_j)$ is *absolutely convergent*, if $\prod_{j=1}^{\infty}(1+|a_j|)$ converges.

Remark. By Theorem 4.3, this is the case if and only if $\sum_{j=1}^{\infty}|a_j|$ converges.

Theorem 4.6. *An absolutely convergent infinite product is convergent.*

Proof. Denote

$$P_n = \prod_{j=1}^{n}(1+a_j) \quad \text{and} \quad Q_n := \prod_{j=1}^{n}(1+|a_j|).$$

Then

$$\begin{aligned} P_n - P_{n-1} &= \prod_{j=1}^{n}(1+a_j) - \prod_{j=1}^{n-1}(1+a_j) \\ &= \Big(\prod_{j=1}^{n-1}(1+a_j)\Big)(1+a_n-1) = a_n \prod_{j=1}^{n-1}(1+a_j) \end{aligned}$$

and, similarly,

$$Q_n - Q_{n-1} = |a_n| \prod_{j=1}^{n-1}(1+|a_j|).$$

Clearly,

$$|P_n - P_{n-1}| \le Q_n - Q_{n-1}.$$

Since $\prod_{j=1}^{\infty}(1+|a_j|)$ converges, $\lim_{n\to\infty} Q_n$ exists. Therefore, $\sum_{j=1}^{\infty}(Q_n - Q_{n-1})$ converges, and so by the standard majorant principle, $\sum_{j=1}^{\infty}(P_n - P_{n-1})$ converges, implying that $\lim_{n\to\infty} P_n$ exists.

It remains to show that this limit is non-zero. Since $\sum_{j=1}^{\infty}|a_j|$ converges, $\lim_{n\to\infty} a_n = 0$, and so $\lim_{n\to\infty}(1+a_n) = 1$. Therefore, $\sum_{j=1}^{\infty}|\frac{a_j}{1+a_j}|$ converges by the majorant principle, since $|1+a_j| \ge \frac{1}{2}$ for j large enough and so $|\frac{a_j}{1+a_j}| \le 2|a_j|$. Therefore

$$\prod_{j=1}^{\infty}\left(1 - \frac{a_j}{1+a_j}\right)$$

is absolutely convergent. By the preceding part of the proof, a finite limit

$$\lim_{n\to\infty} \prod_{j=1}^{n}\left(1 - \frac{a_j}{1+a_j}\right)$$

exists. But

$$\prod_{j=1}^{n}\left(1 - \frac{a_j}{1+a_j}\right) = \prod_{j=1}^{n} \frac{1}{1+a_j} = \frac{1}{\prod_{j=1}^{n}(1+a_j)} = \frac{1}{P_n},$$

and so $\lim_{n\to\infty} P_n \neq 0$. □

Consider finally a sequence $\big(f_j(z)\big)_{j\in\mathbb{N}}$ of analytic functions in a domain $G \subset \mathbb{C}$. Similarly as to Definition 4.2, we say that

$$\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$$

converges in G, if

$$\lim_{n\to\infty} \prod_{j=1}^{n}\big(1+f_j(z)\big) \neq 0$$

exists for each $z \in G$.

Theorem 4.7. *The infinite product* $\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$ *is uniformly convergent in* G, *if the series* $\sum_{j=1}^{\infty}|f_j(z)|$ *converges uniformly in* G.

Proof. Assume

$$\sum_{j=1}^{\infty}|f_j(z)| < M(<\infty)$$

for all $z \in G$. Then by (4.3),

$$\big(1+|f_1(z)|\big)\cdots\big(1+|f_n(z)|\big) \le e^{|f_1(z)|+\cdots+|f_n(z)|} \le e^M.$$

Denote

$$P_n(z) := \prod_{j=1}^{n}\big(1+|f_j(z)|\big).$$

Then

$$P_n(z)-P_{n-1}(z) = |f_n(z)|\big(1+|f_1(z)|\big)\cdots\big(1+|f_n(z)|\big) \le e^M|f_n(z)|.$$

Since

$$\sum_{j=2}^{\infty}\big(P_n(z)-P_{n-1}(z)\big) \le e^M\sum_{j=2}^{\infty}|f_j(z)| \le e^M\sum_{j=1}^{\infty}|f_j(z)|,$$

$\sum_{j=2}^{\infty}\big(P_n(z)-P_{n-1}(z)\big)$ converges uniformly, and so (P_n) as well. This means that $\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$ is absolutely (uniformly) convergent, hence (uniformly) convergent by Theorem 4.6. □

Exercises.

(1) Show that $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{2}{(n+1)(n+2)}\right)=\tfrac{1}{3}$.

(2) Show that $\displaystyle\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}=\tfrac{1}{4}$.

(3) Show that $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$ converges.

(4) Determine whether or not $\displaystyle\prod_{n=0}^{\infty}(1-2^{-n})$ is convergent.

(5) Prove that $\displaystyle\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)$ defines an entire function.

(6) Prove that $\displaystyle\prod_{k=0}^{\infty}(1+z^{2^k})=\frac{1}{1-z}$ for all z in the unit disc $|z|<1$.