5. Weierstrass factorization theorem

Consider a polynomial $P(z)$ with (all) zeros $z_1, \dots, z_n$. Then

$$\begin{aligned} P(z) &= C(z_1 - z)\cdots(z_n - z) \qquad (C \text{ constant}) \\ &= Cz_1\cdots z_n \left(1 - \frac{z}{z_1}\right)\cdots\left(1 - \frac{z}{z_n}\right) \\ &= P(0)\left(1 - \frac{z}{z_1}\right)\cdots\left(1 - \frac{z}{z_n}\right). \end{aligned}$$

Let now $f(z)$ be an entire function with zeros $z_1, z_2, \dots, z_n, \dots$ arranged by increasing moduli, i.e.,

$$0 \le |z_1| \le |z_2| \le \cdots \le |z_n| \le \cdots .$$

By the uniqueness theorem of analytic functions, $\lim_{n\to\infty} |z_m| = \infty$. Assume $z_1 \neq 0$. Then a factorization similar to the polynomial case above is not immediate, since

$$\prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right)$$

may diverge. Therefore, we must somehow modify the situation to ensure the convergence. This may be done by the following

Theorem 5.1. (Weierstraß). *Let $(z_m)_{n\in\mathbb{N}}$ be an arbitrary sequence of complex numbers different from zero, arranged by increasing moduli and $\lim_{n\to\infty} |z_n| = \infty$ and let $m \in \mathbb{N}\cup\{0\}$. Then there exist $\nu \in \mathbb{N}\cup\{0\}$, $\nu = \nu(j)$, such that $\sum_{j=1}^{\infty} |z_j|^{-(\nu+1)}$ converges uniformly in $\mathbb{C}$ and that for the polynomial*

$$Q_\nu(z) := z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{\nu}z^\nu, \qquad \nu \ge 1; \quad Q_0(z) \equiv 0,$$

and for an arbitrary entire function $g(z)$,

$$G(z) := e^{g(z)} z^m \prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right) e^{Q_\nu(\frac{z}{z_j})} \tag{5.1}$$

is an entire function with a zero of multiplicity m at $z = 0$ and with the other zeros exactly at (z_n).

Remark. The sequence (z_n) is *not necessarily* formed by distinct points.

Before proceeding to prove Theorem 5.1, we consider the function (entire)

$$E_\nu(z) := (1 - z)e^{Q_\nu(z)}, \qquad \nu \ge 1; \quad E_0(z) := 1 - z,$$

called usually as the Weierstraß factor.

We first prove three basic properties for $E_\nu(z)$:

(1) $E'_\nu(z) = -z^\nu e^{Q_\nu(z)}$ for $\nu \geq 1$:

$$\begin{aligned} E'_\nu(z) &= -e^{Q_\nu(z)} + (1-z)(1+z+\cdots+z^{\nu-1})e^{Q_\nu(z)} \\ &= e^{Q_\nu(z)}(-1+1+\cdots+z^{\nu-1}-z-z^2-\cdots-z^\nu) = -z^\nu e^{Q_\nu(z)}. \end{aligned}$$

(2) $E_\nu(z) = 1 + \sum_{j>\nu} a_j z^j$ with $\sum_{j>\nu} |a_j| = 1$ for $\nu \geq 0$.

For $\nu = 0$, this is trivial. Since $E_\nu(z)$ is entire, we may consider its Taylor expansion around $z = 0$:

$$E_\nu(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Differentiating, we get

$$\sum_{j=1}^{\infty} j a_j z^{j-1} = E'_\nu(z) = -z^\nu e^{Q_\nu(z)}.$$

Expanding the right hand around $z = 0$, we get $-z^\nu \sum_{j=0}^{\infty} \beta_j z^j$ with $\beta_j \geq 0$ for all j. Therefore $a_1 = a_2 = \cdots = a_\nu = 0$ and $a_j \leq 0$ for $j > \nu$, hence $|a_\nu| = -a_\nu$ for $j > \nu$. Moreover, $a_0 = E_\nu(0) = 1$ and

$$0 = E_\nu(1) = 1 + \sum_{j>\nu} a_j;$$

thus

$$\sum_{j>\nu} a_j = -\sum_{j>\nu} |a_j| = -1,$$

resulting in the assertion.

(3) If $|z| \leq 1$, then $|E_\nu(z) - 1| \leq |z|^{\nu+1}$, $\nu \geq 0$. By (2),

$$\begin{aligned} |E_\nu(z) - 1| &= \Big| \sum_{j=\nu+1}^{\infty} a_j z^j \Big| \leq \sum_{j=\nu+1}^{\infty} |a_j||z|^j \\ &= |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j||z|^{j-(\nu+1)} \leq |z|^{\nu+1} \sum_{j>\nu} |a_j| = |z|^{\nu+1}. \end{aligned}$$

Proof of Theorem 5.1. We consider $E_\nu(\frac{z}{z_j})$ for $j \in \mathbb{N}$. The idea is to determine ν so that $\prod_{j=1}^{\infty} E_\nu(\frac{z}{z_j})$ converges absolutely and uniformly for $|z| < R$, R large enough. To this end, fix $R > 1$ and $0 < \alpha < 1$. Since $\lim_{n\to\infty} |z_m| = \infty$, we find q such that $|z_q| \leq \frac{R}{\alpha}$, while $|z_{q+1}| > \frac{R}{\alpha}$. Then $\prod_{j=1}^{q} E_\nu(\frac{z}{z_j})$ is an entire function as a finite product of entire functions. Consider now the remainder term

$$\prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

in the disc $|z| \leq R$. Since $j > q$, $|z_j| > \frac{R}{\alpha}$ and so

$$|z/z_j| < \alpha < 1.$$

Writing

$$E_\nu\left(\frac{z}{z_j}\right) = \left(1 - \frac{z}{z_j}\right) e^{Q_\nu(\frac{z}{z_j})} = 1 + U_j(z),$$

we proceed to estimate $U_j(z)$. Since $j > q$, and $|z/z_j| < 1$, (3) above implies

$$|U_j(z)| = \left|E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1}. \tag{5.2}$$

We now divide our consideration in two cases:

Case I: There exists $p \in \mathbb{N}$ s.th. $\sum_{j=1}^{\infty} |z_j|^{-p} < \infty$. In this case, we define $\nu := p - 1$. From (4.5), we obtain

$$|U_j(z)| \leq R^p |z_j|^{-p},$$

since $|z| \leq R$. Therefore,

$$\sum_{j=1}^{\infty} |U_j(z)| \leq R^p \sum_{j=1}^{\infty} |z_j|^{-p} < \infty$$

for $|z| \leq R$. By Theorem 4.3 and Definition 4.6,

$$\prod_{j=q+1}^{\infty} \big(1 + U_j(z)\big) = \prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely and uniformly.

Case II: For all $p \in \mathbb{N}$, $\sum_{j=1}^{\infty} |z_j|^{-p} = \infty$. In this case, we take $\nu = j - 1$, so ν depends on j. Then, by (5.2) again

$$|U_j(z)| \leq \left|\frac{z}{z_j}\right|^j$$

provided $j > q$ (which means $|\frac{z}{z_j}| < \alpha < 1$) and $|z| \leq R$. Since $|z/z_j| < \alpha < 1$, we have

$$\limsup_{j\to\infty} \sqrt[j]{\left|\frac{z}{z_j}\right|^j} \leq \alpha < 1,$$

and therefore, by the root test, which carries over from the (real) analysis word by word, $\sum_{j=q+1}^{\infty} |U_j(z)|$ converges. As above, we get that $\prod_{j=q+1}^{\infty} E_\nu(\frac{z}{z_j})$ converges absolutely and uniformly for $|z| \leq R$. *If we now have proved* that $\prod_{j=1}^{\infty} E_\nu(\frac{z}{z_j})$ is analytic in $\mathbb{C}$, then $G(z)$ is entire and has exactly the desired zeros. Therefore, it remains to prove

Theorem 5.2. *If $\big(f_n(z)\big)$ is a sequence of analytic functions in a domain G and if there exists*

$$\lim_{n\to\infty} f_n(z) = f(z) \tag{5.3}$$

uniformly in closed subdomains of G, then $f(z)$ is analytic and $f'(z) = \lim_{n\to\infty} f_n'(z)$.

Proof. This is a consequence of the Cauchy integral formula. In fact, fix $z \in G$ arbitrarily and let $B(z,r)$ be a disc s.th. $\overline{B(z,r)} \subset G$. By the Cauchy formula,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta, \qquad n \in \mathbb{N}.$$

Since the convergence is uniform on ∂B,

$$|f_n(\zeta) - f(\zeta)| < \varepsilon$$

for $n \geq n_\varepsilon$ and for all $\zeta \in \partial B$. Therefore,

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\partial B}\frac{f_n(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{\zeta - z}\,d\zeta\right| \\
&\qquad \leq \frac{1}{2\pi}\int_{\partial B}\frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|}\,|d\zeta| \leq \frac{\varepsilon \cdot 2\pi r}{2\pi \cdot r} = \varepsilon,
\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial B}\frac{f_n(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial B}\int_{\partial B}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

By (5.3),

$$f(z) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Now, $f'(z)$ exists, since

$$\begin{aligned}
\frac{1}{h}[f(z+h) - f(z)] &= \frac{1}{2\pi h i}\int_{\partial B}\left(\frac{f(\zeta)}{\zeta - (z+h)} - \frac{f(\zeta)}{\zeta - z}\right)d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta - z)(\zeta - (z+h))}\,d\zeta \to \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta.
\end{aligned}$$

Therefore, $f(z)$ is analytic. Since the limit (5.3) is uniform in ∂B, we get

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta = \frac{1}{2\pi i}\int_{\partial B}\big(\lim_{n\to\infty} f_n(\zeta)\big)\frac{d\zeta}{(\zeta - z)^2} \\
&= \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial B}\frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta. \quad \square
\end{aligned}$$

Theorem 5.3. (Weierstraß product theorem). *Let $f(z)$ be entire with a zero of multiplicity $m \in \mathbb{N} \cup \{0\}$ at $z = 0$ and the zeros $z_j \neq 0$ s.th. $0 < |z_1| \leq |z_2| \leq \cdots$, possibly including repeated points. Let $H(z)$ denote the Weierstraß product* (5.1) *with $g(z) \equiv 0$. Then there exists an entire function $h(z)$ s.th.*

$$f(z) = H(z)e^{h(z)}. \tag{5.4}$$

Proof. Since $f(z)$ and $H(z)$ have exactly the same zeros, it is clear that $f(z)/H(z)$ is entire with no zeros. Applying Theorem 4.1 results in (5.4)

Observe that Theorem 5.1 may be expressed as

Theorem 5.4. *Let* $(z_n)_{n\in\mathbb{N}}$ *be a sequence the distinct complex numbers having no finite accumulation points, and let a sequence* $(k_n)_{n\in\mathbb{N}}$ *of natural numbers be given. Then there exists an entire function having roots of multiplicity* k_n *at* z_n *for all* $n \in \mathbb{N}$*, and nowhere else.*