6. Complex interpolation

This section is entirely devoted to prove the following interpolation theorem for analytic functions:

Theorem 6.1. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of distinct points in $\mathbb{C}$ having no finite accumulation points and $(\zeta_n)_{n\in\mathbb{N}}$ a sequence of complex numbers, not necessarily distinct. Then there exists an entire function $f(z)$ such that $f(z_n) = \zeta_n$ for all $n \in \mathbb{N}$.*

To prove this result, we first need to prove the following Mittag-Leffler theorem. To this end, recall that Definition 3.1 for a meromorphic function f. By this definition, the Laurent expansion of f around $a \in \mathbb{C}$ must be of the form

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-a)^j,$$

where $m = m(a)$. The finite part

$$\sum_{j=-m}^{-1} a_j (z-a)^j$$

is called *the singular part* of f at $z = a$.

Theorem 6.2. (Mittag-Leffler). *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of distinct points in $\mathbb{C}$ having no finite accumulation points, and let $\big(P_n(z)\big)_{n\in\mathbb{N}}$ be a sequence of polynomials such that $P_n(0) = 0$. Then there exists a meromorphic function $f(z)$ having the singular part*

$$P_n\left(\frac{1}{z-z_n}\right)$$

at $z = z_n$, and no other poles in $\mathbb{C}$.

Proof. We may assume that $|z_1| \le |z_2| \le \cdots$. Moreover, we assume, temporarily, that $b_1 \neq 0$. Next, let $\sum_{n=1}^{\infty} c_n$ be a convergent series of strictly positive real numbers. As $P_n(z)$ is a polynomial, $P_n(\frac{1}{z-z_n})$ must be analytic in $B(0, |b_n|)$; therefore we may take its Taylor expansion

$$P_n\left(\frac{1}{z-z_n}\right) = \sum_{j=0}^{\infty} a_j^{(n)} z^j \tag{6.1}$$

in $B(0, |b_n|)$. By elementary facts of (complex) power series, (6.1) converges absolutely and uniformly in $B(0, \rho)$, where $|z_n|/2 < \rho < |z_n|$. Denote now

$$Q_n(z) := \sum_{j=0}^{k_n} a_j^{(n)} z^j, \tag{6.2}$$

where k_n has been chosen large enough to satisfy

$$\sup_{z\in B(0,\frac{|z_n|}{2})} \left| P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right| < c_n. \tag{6.3}$$

We now proceed to consider the series

$$\sum_{n=1}^{\infty} \left(P_n \left(\frac{1}{z - z_n} \right) - Q_n(z) \right). \tag{6.4}$$

Take now an arbitrary $R > 0$. Clearly, only those singular parts $P_n\big(1/(z - z_n)\big)$ with $z_n \in B(0, R)$ contribute poles to the sum (6.4). We now break the sum (6.4) in two parts:

$$\sum_{|z_n| \le 2R} \left(P_n \left(\frac{1}{z - z_n} \right) - Q_n(z) \right) \sum_{|z_n| > 2R} \left(P_n \left(\frac{1}{z - z_n} \right) - Q_n(z) \right). \tag{6.5}$$

The second (infinite) part has no poles in $B(0, R)$. Moreover, in this part, $R < |z_n|/2$, and so, by (6.3),

$$\sup_{z \in \overline{B(0,R)}} \left| P_n \left(\frac{1}{z - z_n} \right) - Q_n(z) \right| < c_n.$$

By the standard majorant principle, the infinite part of (6.5) converges absolutely and uniformly in $\overline{B(0, R)}$, and therefore defines an analytic function in $B(0, R)$. The first part in (6.5) is a rational function with prescribed behavior of poles exactly at $z = z_n \in B(0, R)$.

Now, since R is arbitrary, the series (6.4) converges locally uniformly in $\mathbb{C} \setminus \bigcup_{n=1}^{\infty} \{z_n\}$, having prescribed behavior of poles in $\mathbb{C}$ except perhaps at $z = 0$. Adding one singular part, say $P_0(1/z)$, for $z = 0$, we obtain a function with the asserted properties.

Proof of Theorem 6.1. By Theorem 5.4, construct an entire function $g(z)$ with simple zeros only, exactly at each z_n. Then $g'(z_n) \neq 0$ for all $n \in \mathbb{N}$. By the Mittag-Leffler theorem, there exists a meromorphic function $h(z)$ with simple poles only exactly at each z_n, with residue $\zeta_n / g'(z_n)$ at each z_n. Consider $f(z) := h(z)g(z)$, analytic except perhaps at the points z_n. But near $z = z_n$,

$$\begin{aligned} g(z) &= g'(z_n)(z - z_n) + \cdots = (z - z_n) g_n(z), \qquad & g_n(z_n) &= g'(z_n) \\ h(z) &= \frac{\zeta_n}{g'(z_n)} \cdots \frac{1}{z - z_n} + \cdots = \frac{h_n(z)}{z - z_n}, \qquad & h_n(z_n) &= \frac{\zeta_n}{g'(z_n)}, \end{aligned}$$

where $g_n(z)$, $h_n(z)$ are analytic at $z = z_n$. Therefore, $f(z) = g_n(z)h_n(z)$ near $z = z_n$, and so analytic. Moreover,

$$f(z_n) = g_n(z_n) h_n(z_n) = g'(z_n) \cdots \frac{\zeta_n}{g'(z_n)} = \zeta_n$$

for each z_n. $\square$