7. Growth of entire functions

Definition 7.1. For an entire function $f(z)$,

$$M(r, f) = \max_{|z| \le r} |f(z)|$$

is the *maximum modulus* of f.

Remark. By the maximum principle,

$$M(r, f) = \max_{|z| = r} |f(z)|.$$

Lemma 7.2. *Let $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, be a polynomial. Given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ s.th.*

$$(1-\varepsilon)|a_n| r^n \le |P(z)| \le (1+\varepsilon)|a_n| r^n$$

whenever $r = |z| > r_\varepsilon$.

Proof. Clearly, $|P(z)| = |a_n||z|^n \left| 1 + \frac{a_{n-1}}{a_n} \frac{1}{z} + \cdots + \frac{a_0}{a_n} \frac{1}{z^n} \right|$. Denote

$$r_n(z) = \frac{a_{n-1}}{a_n} \frac{1}{z} + \cdots + \frac{a_0}{a_n} \frac{1}{z^n}.$$

Obviously, $|r_n(z)| < \varepsilon$, if $|z| > r_\varepsilon$ for some $\varepsilon > 0$. This means that

$$\begin{aligned}(1-\varepsilon)|a_n| r^n &\le \big(1 - |r_n(z)|\big)|a_n| r^n \le |1 + r_n(z)||a_n| r^n \\ &= |P(z)| \le \big(1 + |r_n(z)|\big)|a_n| r^n \le (1+\varepsilon)|a_n| r^n. \quad \square\end{aligned}$$

Definition 7.3. For an entire function $f(z)$, the *order*, resp. *lower order*, is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \qquad \text{resp.} \quad \mu(f) := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Remark. By the Liouville theorem, $\rho(f) \ge 0$ and $\mu(f) \ge 0$.

Examples. (1) Show that $\rho(e^z) = 1 = \mu(e^z)$.
(2) For a polynomial $P(z)$, show that $\rho(P) = \mu(P) = 0$.
(3) Determine $\rho(\cos z)$.
(4) Consider

$$f(z) = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots \qquad (= \cos\sqrt{z}).$$

Show that f is entire and determine $\rho(f)$.

Definition 7.4. Given an entire function $f(z)$, define

$$A(r, f) := \max_{|z|=r} \operatorname{Re} f(z).$$

Theorem 7.5. *For an entire function* $f(z) = \sum_{j=0}^{\infty} a_j z^j$,

$$|a_j| r^j \leq \max[0, 4A(r,f)] - 2 \operatorname{Re} f(0), \tag{7.1}$$

for all $j \in \mathbb{N}$.

Proof. For $r = 0$, the assertion is trivial. So, assume $r > 0$, and denote $z = re^{i\varphi}$, $a_n = \alpha + i\beta_n$. Then

$$\begin{aligned}
\operatorname{Re} f(re^{i\varphi}) &= \operatorname{Re} \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j (\cos\varphi + i \sin\varphi)^j \\
&= \operatorname{Re} \sum_{j=0}^{\infty} (\alpha_j + i\beta_j)(\cos j\varphi + i \sin j\varphi) r^j \\
&= \sum_{j=0}^{\infty} (\alpha_j \cos j\varphi - \beta_j \sin j\varphi) r^j.
\end{aligned}$$

Multiply now by $\cos n\varphi$, resp. by $\sin n\varphi$, and integrate term by term. This results in

$$\begin{aligned}
\alpha_n r^n &= \frac{1}{\pi} \int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big) \cos n\varphi \, d\varphi, \qquad n > 0, \\
-\beta_n r^n &= \frac{1}{\pi} \int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big) \sin n\varphi \, d\varphi, \qquad n > 0, \\
\alpha_0 &= \frac{1}{2\pi} \int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big) \, d\varphi, \qquad \beta_0 = 0.
\end{aligned}$$

Subtracting for $n > 0$, we obtain

$$\begin{aligned}
a_n r^n &= (\alpha_n + i\beta_n) r^n \\
&= \frac{1}{\pi} \int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)(\cos n\varphi - i \sin n\varphi) \, d\varphi \\
&= \frac{1}{\pi} \int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big) e^{-in\varphi} \, d\varphi,
\end{aligned}$$

and so

$$\begin{aligned}
|a_n| r^n &\leq \frac{1}{\pi} \int_0^{2\pi} |\operatorname{Re} f(re^{i\varphi})| \, d\varphi, \\
|a_n| r^n + 2\alpha_0 &\leq \frac{1}{\pi} \int_0^{2\pi} \big(|\operatorname{Re} f(re^{i\varphi})| + \operatorname{Re} f(re^{i\varphi})\big) \, d\varphi. \qquad (7.2)
\end{aligned}$$

If $A(r,f) < 0$, then $|\operatorname{Re} f(re^{i\varphi})| + \operatorname{Re} f(re^{i\varphi}) = 0$, and (7.1) is an immediate consequence of (7.2). If $A(r,f) \geq 0$, then

$$|a_n| r^n + 2\alpha_0 \leq \frac{1}{\pi} \int_0^{2\pi} 2A(r,f) \, d\varphi = 4A(r,f);$$

the proof is now complete. □

Theorem 7.6. (Hadamard). *If $f(z)$ is entire and*

$$L := \liminf_{r\to\infty} A(r,f)r^{-s} < \infty$$

for some $s \geq 0$, then $f(z)$ is a polynomial of degree $\deg f \leq s$.

Proof. By assumption, there is a sequence $r_n \to \infty$ such that $A(r_n, f) \leq (L+1)r_n^s$. If now $j > s$, then

$$|a_j| r_n^j \leq 4(L+1)r_n^s - 2\operatorname{Re} f(0)$$

by Theorem 7.5. Therefore

$$|a_j| \leq \frac{4(L+1)}{r_n^{j-s}} - \frac{2\operatorname{Re} f(0)}{r_n^j} \to 0 \qquad \text{as } r_n \to \infty.$$

So, $a_j = 0$ for all $j > s$. □

Theorem 7.7. *Let $f(z)$ be entire with no zeros such that $\mu(f) < \infty$. Then $f(z) = e^{P(z)}$ for a polynomial*

$$P(z) = a_m z^m + \cdots + a_0, \qquad a_n \neq 0,$$

such that $m = \mu(f) = \rho(f)$.

Proof. By Theorem 4.1, $f(z) = e^{g(z)}$ for an entire function $g(z)$. Now, given $\varepsilon > 0$, there is a sequence $r_n \to \infty$ such that for any z with $|z| = r_n$,

$$e^{\operatorname{Re} g(z)} = |e^{g(z)}| = |f(z)| \leq e^{r_n^{\mu(f)+\varepsilon}}. \tag{7.3}$$

From the definition of the lower order,

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \mu(f),$$

it follows that

$$\log\log M(r,f) \leq \big(\mu(f)+\varepsilon\big)\log r,$$

and so

$$M(r,f) \leq e^{r^{\mu(f)+\varepsilon}}.$$

By (7.3), $\operatorname{Re} g(z) \leq r_n^{\mu(f)+\varepsilon}$ for all $|z| = r_n$, hence

$$A(r_n, g) \leq r_n^{\mu(f)+\varepsilon}.$$

By Theorem 7.6,

$$\liminf_{r\to\infty} A(r,g)r^{-(\mu(f)+\varepsilon)} \leq 1 < \infty,$$

and so, g must be a polynomial of degree $\leq \mu(f)+\varepsilon$, hence $\leq \mu(f)$.

We still have to prove that $\mu(f) = \rho(f) = m$ for $f(z) = e^{P(z)}$, if $P(z) = a_m z^m + \cdots + a_0$, $a_m \neq 0$.

To this end, we first observe, by Lemma 7.2, that

$$|f(z)| = |e^{P(z)}| = e^{\operatorname{Re} P(z)} \leq e^{|P(z)|} \leq e^{2|a_m|r^m}$$

for every $|z| = r$, r sufficiently large. Therefore,

$$\log M(r,f) \leq 2|a_m| r^m,$$
$$\log\log M(r,f) \leq m \log r + \log(2|a_m|)$$

and so

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \leq \limsup_{r\to\infty} \frac{m\log r + \log(2|a_m|)}{\log r} = m.$$

So,

$$\rho(f) \leq m = \deg P \leq \mu(f) \leq \rho(f),$$

and we are done. □

Now, let $f(z)$ be an entire function of finite order $\rho < +\infty$. By the definition of the order, this means that for some r_ε,

$$\frac{\log\log M(r,f)}{\log r} < \rho + \varepsilon, \qquad \text{for all } r \geq r_\varepsilon,$$

hence

$$\log\log M(r,f) < (\rho+\varepsilon)\log r = \log r^{\rho+\varepsilon}$$

and so

$$|f(z)| \leq M(r,f) \leq e^{r^{\rho+\varepsilon}} \qquad \text{for all } |z| \leq r. \tag{7.4}$$

Lemma 7.8. *Defining*

$$\alpha := \inf\{\, \lambda > 0 \mid M(r,f) \leq e^{r^\lambda} \text{ for all } r \text{ suff. large}\},$$

the order of f satisfies $\rho(f) = \alpha$.

Proof. By (7.4), $\alpha \leq \rho(f) + \varepsilon$ for all $\varepsilon > 0$, so $\alpha \leq \rho(f)$. On the other hand, given any $\lambda > 0$ such that the condition is satisfied, we get

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \leq \limsup_{r\to\infty} \frac{\log\log e^{r^\lambda}}{\log r} = \lambda$$

and so $\rho(f) \leq \alpha$. □

Theorem 7.9. *Let $f_1(z)$, $f_2(z)$ be two entire functions. Then*

(1) $\rho(f_1 + f_2) \leq \max\big(\rho(f_1), \rho(f_2)\big)$,
(2) $\rho(f_1 f_2) \leq \max\big(\rho(f_1), \rho(f_2)\big)$.

Moreover, if $\rho(f_1) < \rho(f_2)$, then

(3) $\rho(f_1 + f_2) = \rho(f_2)$,

Proof. (1) Assume therefore that $\rho(f_1) = \rho(f_2) = \rho$. By Lemma 7.8, for r sufficiently large,

$$M(r,f_1) \leq e^{r^{\rho+\varepsilon}}, \qquad M(r,f_2) \leq e^{r^{\rho+\varepsilon}}.$$

By elementary estimates, for r sufficiently large,

$$\begin{aligned}M(r,f_1+f_2) &= \max_{|z|=r}|f(z_1)+f(z_2)| \le \max_{|z|=r}|f(z_1)|+\max_{|z|=r}|f(z_2)| \\ &= M(r,f_1)+M(r,f_2) \le e^{r^{\rho_1+\varepsilon}}+e^{r^{\rho_2+\varepsilon}} \le 2e^{r^{\max(\rho_1,\rho_2)+\varepsilon}} \\ &\le e^{r^{\max(\rho_1,\rho_2)+2\varepsilon}}.\end{aligned}$$

By Lemma 7.8 again, $\rho(f_1+f_2)\le\rho+2\varepsilon$ and so $\rho(f_1+f_2)\le\rho$.

(2) Similarly, for $\rho_1=\rho(f_1)$, $\rho_2=\rho(f_2)$,

$$\begin{aligned}M(r,f_1f_2) &= \max_{|z|=r}|f_1(z)f_2(z)| \le \big(\max_{|z|=r}|f_1(z)|\big)\big(\max_{|z|=r}|f_2(z)|\big) \\ &= M(r,f_1)M(r,f_2) \le e^{r^{\rho_1+\varepsilon}}\cdot e^{r^{\rho_2+\varepsilon}} \le e^{r^{\max(\rho_1,\rho_2)+\varepsilon}}\end{aligned}$$

and we obtain $\rho(f_1f_2)\le\max\big(\rho(f_1),\rho(f_2)\big)$ by taking logarithms twice.

(3) We now assume $\rho(f_1)<\rho(f_2)=\rho$. The inequality in (1) is immediate:

$$M(r,f_1+f_2)\le M(r,f_1)+M(r,f_2)\le e^{r^{\rho(f_1)+\varepsilon}}+e^{r^{\rho+\varepsilon}}\le 2e^{r^{\rho+\varepsilon}}\le e^{r^{\rho+2\varepsilon}}.$$

Therefore, it remains to prove that for any $\varepsilon>0$,

$$\rho(f_1+f_2)\ge\rho-\varepsilon.$$

Now, we again have $M(r,f_1)\le e^{r^{\rho(f_1)+\varepsilon}}$ for all r sufficiently large and, by the definition of lim sup,

$$M(r,f_2)\ge e^{r_n^{\rho-\varepsilon}} \tag{7.5}$$

for a sequence (r_n) such that $r_n\to\infty$ as $n\to\infty$. Now, given r_n, since f_2 is continuous and $|z|=r_n$ is compact, we find z_n such that $|z_n|=r_n$ and that $|f(z_n)|=M(r_n,f_2)\ge\exp(r_n^{\rho-\varepsilon})$ by (7.5). Therefore

$$|(f_1+f_2)(z_n)|=|f_1(z_n)+f_2(z_n)|\ge|f_2(z_n)|-|f_1(z_n)|\ge e^{r_n^{\rho-\varepsilon}}-e^{r_n^{\rho(f_1)+\varepsilon}}.$$

To estimate further, take $\varepsilon>0$ so that $\rho-\varepsilon>\rho(f_1)+\varepsilon>0$. Then

$$r_n^{\rho(f_1)+\varepsilon}-r_n^{\rho-\varepsilon}=r_n^{\rho-\varepsilon}(r_n^{\rho(f_1)-\rho+2\varepsilon}-1)\to-\infty$$

as $n\to\infty$, since $\rho(f_1)-\rho<0$. Therefore,

$$\begin{aligned}M(r_n,f_1+f_2)\ge|(f_1+f_2)(z_n)| &\ge e^{r_n^{\rho-\varepsilon}}-e^{r_n^{\rho(f_1)+\varepsilon}} \\ &= e^{r_n^{\rho-\varepsilon}}(1-e^{r_n^{\rho(f_1)+\varepsilon}-r_n^{\rho-\varepsilon}})\ge\tfrac{1}{2}e^{r_n^{\rho-\varepsilon}}\end{aligned}$$

for n sufficiently large, since $e^{r_n^{\rho(f_1)+\varepsilon}-r_n^{\rho-\varepsilon}}\to0$ as $n\to\infty$. □

Remark. If $\rho(f_1) < \rho(f_2)$, then $\rho(f_1 f_2) = \rho(f_2)$ also holds. This can be proved with some more knowledge on meromorphic functions. In fact, since $1/f_1$ is meromorphic and non-entire in general, and so we cannot directly apply the above reasoning.

Considering an entire function f with the Taylor expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

it is possible to determine its order by the coefficients a_j.

Theorem 7.10. *Defining*

$$b_j := \begin{cases} 0, & \text{if } a_j = 0 \\ \frac{j \log j}{\log \frac{1}{|a_j|}}, & \text{if } a_j \neq 0, \end{cases}$$

the order $\rho(f)$ of f is determined by

$$\rho(f) = \limsup_{j \to \infty} b_j.$$

Proof. Denote $\mu := \limsup_{j\to\infty} b_j$.

1) We first prove that $\rho(f) \geq \mu$. If $\mu = 0$, this inequality is trivial. So, we may assume $\mu > 0$. Recall first Cauchy inequalities:

$$\begin{aligned} |a_j| &= \Big| \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta^{j+1}} \Big| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta|^{j+1}} r \, d\varphi \\ &\leq \frac{M(r,f)}{2\pi} \int_0^{2\pi} r^{-j} \, d\varphi = \frac{M(r,f)}{r^j}, \qquad \text{for all } j \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Take now $\sigma \in \mathbb{R}$ such that $0 < \sigma < \mu$, and proceed to prove that $\rho(f) \geq \sigma$. Since σ is arbitrary, this means that $\rho(f) \geq \mu$. By the definition of σ and μ, there exist infinitely many natural numbers j such that

$$j \log j \geq \sigma \log \frac{1}{|a_j|} = -\sigma \log |a_j|$$

$\Longrightarrow$

$$\log |a_j| \geq -\frac{1}{\sigma} j \log j.$$

By the Cauchy inequalities,

$$\log M(r,f) \geq \log(r^j |a_j|) = j \log r + \log |a_j| \geq j \log r - \frac{1}{\sigma} j \log j.$$

The above j:s will be used to determine a sequence of r-values as follows:

$$r_j := (ej)^{1/\sigma}, \qquad \text{hence } j = \frac{1}{e} r_j^{\sigma}.$$

Then

$$\log M(r_j, f) \geq j \cdot \frac{1}{\sigma}\log(ej) - \frac{1}{\sigma} j\log j = \frac{1}{\sigma} j = \frac{1}{\sigma e} r_j^{\sigma}$$

$\Longrightarrow$

$$\log\log M(r_j, f) \geq \sigma \log r_j + \log \frac{1}{\sigma e}$$

$\Longrightarrow$

$$\begin{aligned}\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} &\geq \limsup_{r_j\to\infty} \frac{\log\log M(r_j,f)}{\log r_j} \\ &\geq \limsup_{r_j\to\infty} \frac{\sigma \log r_j + \log\frac{1}{\sigma e}}{\log r_j} = \sigma.\end{aligned}$$

2) To prove that $\sigma(f) \leq \mu$, we may now assume that $\mu < +\infty$. Fix $\varepsilon > 0$. Then, for all sufficiently large j, such that $a_j \neq 0$,

$$0 \leq \frac{j\log j}{\log\frac{1}{|a_j|}} \leq \mu + \varepsilon.$$

Therefore,

$$\frac{j}{\mu+\varepsilon}\log j \leq \log\frac{1}{|a_j|} = -\log|a_j|$$

and so

$$\log|a_j| \leq -\frac{j}{\mu+\varepsilon}\log j = \log(j^{-\frac{j}{\mu+\varepsilon}}).$$

By monotonicity of the logarithm,

$$|a_j| \leq j^{-j/(\mu+\varepsilon)}.$$

Now,

$$\begin{aligned}M(r,f) = \max_{|z|=r}\Big|\sum_{j=0}^{\infty} a_j z^j\Big| &\leq |a_0| + \sum_{j=1}^{\infty}|a_j| r^j \leq |a_0| + \sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}} r^j \\ &= |a_0| + \sum_{0\neq j<(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j + \sum_{j\geq(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j \\ &= S_1 + S_2 + |a_0|.\end{aligned}$$

Since $(2r)^{\mu+\varepsilon} \leq j$ in the sum S_2, we get

$$2r \leq j^{\frac{1}{\mu+\varepsilon}},$$

hence $rj^{-\frac{1}{\mu+\varepsilon}} \leq \frac{1}{2}$ and so

$$S_2 = \sum_{j\geq(2r)^{\mu+\varepsilon}} (rj^{-\frac{1}{\mu+\varepsilon}})^j \leq \sum_{j\geq(2r)^{\mu+\varepsilon}} \left(\tfrac{1}{2}\right)^j \leq \sum_{j=1}^{\infty}\left(\tfrac{1}{2}\right)^j \leq 2.$$

For S_1, we obtain

$$S_1 = \sum_{0 \neq j < (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j \le \sum_{0 \neq j < (2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^{(2r)^{\mu+\varepsilon}}$$
$$\le r^{(2r)^{\mu+\varepsilon}} \sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}} = K r^{(2r)^{\mu+\varepsilon}}, \qquad K < \infty.$$

In fact, since

$$j^{-\frac{j}{\mu+\varepsilon}} \le \frac{1}{j^2}$$

for all j sufficiently large, the sum $\sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}}$ converges. Therefore,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \le \limsup_{r\to\infty} \frac{\log\log(S_1 + S_2 + |a_0|)}{\log r}$$
$$= \limsup_{r\to\infty} \frac{\log\log S_1}{\log r} \le \limsup_{r\to\infty} \frac{\log\log(K r^{(2r)^{\mu+\varepsilon}})}{\log r}$$
$$\le \mu + 2\varepsilon$$

and so

$$\rho(f) \le \mu. \quad \square$$

Example. Consider

$$f(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j,$$

and recall the Stirling formula

$$\lim_{j\to\infty} \left(j! / \sqrt{2\pi j} e^{-j} j^j \right) = 1.$$

Now,

$$\frac{1}{b_j} = \frac{\log(j!)}{j \log j} \sim \frac{j \log j - j + \log\sqrt{2\pi j}}{j \log j} \to 1$$

and so $\rho(e^z) = \limsup_{j\to\infty} b_j = 1$, as already known.

Definition 7.11. For an entire function $f(z)$ of order ρ such that $0 < \rho < \infty$, its type τ is defined by

$$\tau = \tau(f) := \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho}.$$

The next lemma is a counterpart to Lemma 7.8:

Lemma 7.12. *Define*

$$\beta := \inf\{ K > 0 \mid M(r,f) \le e^{Kr^\rho} \textit{ for all } r \textit{ sufficiently large} \},$$

where f is entire and $\rho = \rho(f)$, $\rho \in (0, +\infty)$. Then $\beta = \tau(f)$.

Proof. Observe that we understand, as usually, that $\inf \Phi = +\infty$.

1) If $\tau(f) = +\infty$, then for all $K > 0$, there is a sequence $r_n \to \infty$ such that

$$\log M(r_n, f) \geq K r_n^\rho$$

and so

$$M(r_n, f) \geq \exp(K r_n^\rho).$$

Therefore, there is no $K > 0$ such that

$$M(r, f) \leq e^{K r^\rho}$$

for all r sufficiently large, implying that

$$\beta = +\infty.$$

Conversely, if $\beta = +\infty$, then $\{\, K > 0 \mid M(r, f) \leq e^{K r^\rho}$ for all r sufficiently large $\}$ $= \Phi$. So, for all $K > 0$, we find a sequence $r_n \to +\infty$ such that $M(r_n, f) > \exp(K r_n^\rho)$. Therefore $\tau(f) = +\infty$.

2) Take now K $(\geq \beta)$ such that $M(r, f) \leq e^{K r^\rho}$ for all r sufficiently large. But then

$$\frac{\log M(r, f)}{r^\rho} \leq \frac{K r^\rho}{r^\rho} = K$$

for all r sufficiently large. This results in

$$\tau(f) = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho} \leq K.$$

Since $K \geq \beta$ is arbitrary, we conclude that $\tau(f) \leq \beta$.

3) To prove that $\tau(f) \geq \beta$, observe, by the definition of $\tau(f)$, that given $\varepsilon > 0$,

$$\frac{\log M(r, f)}{r^\rho} \leq \tau(f) + \varepsilon$$

for all r sufficiently large. Then

$$\log M(r, f) \leq \big(\tau(f) + \varepsilon\big) r^\rho$$

and so

$$M(r, f) \leq \exp\big(\big(\tau(f) + \varepsilon\big) r^\rho\big).$$

This implies

$$\beta \leq \tau(f) + \varepsilon,$$

hence

$$\beta \leq \tau(f). \quad \square$$

Lemma 7.13. *Let $f(z)$ be analytic in a neighborhood of $z = 0$ with the Taylor expansion*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j. \tag{7.6}$$

Suppose there exist $\lambda > 0$, $\mu > 0$ and a natural number $N = N(\mu, \lambda) > 0$ such that

$$|a_j| \le (e\mu\lambda/j)^{j/\mu} \tag{7.7}$$

for all $j > N$. Then the Taylor expansion converges in the whole complex plane, and therefore $f(z)$ is entire. Moreover, for every $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$M(r, f) \le e^{(\lambda+\varepsilon)r^{\mu}}$$

for all $r > R$.

Proof. By (7.7),

$$\sqrt[j]{|a_j|} \le \left(\frac{e\mu\lambda}{j}\right)^{1/\mu} \to 0 \qquad \text{as } j \to \infty.$$

Therefore, the radius of convergence R for the power series (7.8) is $R = +\infty$, since

$$\frac{1}{R} = \limsup_{j\to\infty} \sqrt[j]{|a_j|} = 0.$$

Therefore, (7.6) determines an entire function.

To prepare the subsequent estimate for $M(r, f)$, observe first (exercise!) that the maximum of

$$\left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x$$

for $x \ge 0$ will be achieved as $x = \mu\lambda r^{\mu}$. Therefore,

$$\left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x \le e^{\lambda r^{\mu}}.$$

Moreover, if $j > N(r) := \max(N, 2^{\mu} e\mu\lambda r^{\mu})$, then

$$\sqrt[j]{|a_j| r^j} < \left(\frac{e\mu\lambda}{j}\right)^{1/\mu} r < \tfrac{1}{2},$$

and so

$$|a_j| r^j < \frac{1}{2^j} \qquad \text{for } j > N(r).$$

For the maximum modulus of f, we now obtain, for $r > 1$,

$$
\begin{aligned}
M(r,f) &= \max_{|z|=r}\Big|\sum_{j=0}^{\infty} a_j z^j\Big| \le \sum_{j=0}^{\infty} |a_j| r^j \\
&= \sum_{j=0}^{N} |a_j| r^j + \sum_{j=N+1}^{N(r)} |a_j| r^j + \sum_{j=N(r)+1}^{\infty} |a_j| r^j \\
&\le r^N\Big(\sum_{j=0}^{N} |a_j|\Big) + \big(N(r)-N\big) \max_{N+1\le j\le N(r)} |a_j| r^j + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&\le r^N\Big(\sum_{j=0}^{N} |a_j|\Big) + \big(N(r)-N\big) \max_{j>N} (|a_j| r^j) + 1 \\
&\le r^N\underbrace{\Big(\sum_{j=0}^{N} |a_j|\Big)}_{=:b} + \big(N(r)-N\big) \max_{j\ge N} \left(\left(\frac{e\mu\lambda}{j}\right)^{j/\mu} r^j\right) + 1 \\
&\le 1 + br^N + \max(0, 2^\mu e\mu\lambda r^\mu - N) e^{\lambda r^\mu} \le e^{(\lambda+\varepsilon) r^\mu},
\end{aligned}
$$

provided r is sufficiently large. □

Theorem 7.14. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of finite order $\rho > 0$ and of type $\tau = \tau(f)$. Then*

$$\tau = \frac{1}{e\rho} \limsup_{j\to\infty} (j |a_j|^{\rho/j}).$$

Proof. Denoting $\nu := \limsup_{j\to\infty}(j|a_j|^{\rho/j})$, we have to prove that $\tau = \frac{\nu}{e\rho}$.

1) We first prove that $\tau \le \nu/e\rho$. If $\nu = +\infty$, this is trivial. Therefore, we may assume that $(0 \le)\nu < +\infty$. Take any $K > \nu/e\rho$, i.e. $e\rho K > \nu$. By the definition of ν,

$$j|a_j|^{\rho/j} < e\rho K$$

for j sufficiently large. Hence,

$$|a_j| < \left(\frac{e\rho K}{j}\right)^{j/\rho}.$$

By Lemma 7.13, for each $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that

$$M(r,f) \le e^{(K+\varepsilon) r^\rho}$$

whenever $r > R(\varepsilon)$. by Definition 7.11, $\tau \le K + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\tau \le K$ and since $K > \nu/e\rho$ is arbitrary,

$$(0 \le)\tau \le \nu/e\rho. \tag{7.8}$$

2) To prove the reverse inequality, we first observe that $\nu = 0$ implies $\tau = 0$ by (7.8), so we may now assume that $0 < \nu \leq +\infty$. Take β such that $0 < \beta < \nu$. By the definition of ν again, there is a sequence of j:s ($\to \infty$) such that

$$j|a_j|^{\rho/j} \geq \beta$$

and so

$$|a_j| \geq (\beta/j)^{j/\rho}.$$

Corresponding to these j:s define a sequence r_j by

$$(r_j)^\rho = je/\beta \to \infty \qquad \text{as } j \to \infty. \tag{7.9}$$

By the Cauchy inequalities $|a_j| \leq \frac{M(r,f)}{r^j}$, we obtain by (7.9)

$$M(r_j, f) \geq |a_j|(r_j)^j \geq \left(\frac{\beta}{j}\right)^{j/\rho} \left(\frac{je}{\beta}\right)^{j/\rho} = e^{j/\rho} \stackrel{(*)}{=} e^{\frac{1}{\rho}\frac{\beta}{e}(r_j)^\rho}.$$

Therefore,

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho} \geq \limsup_{j\to\infty} \frac{\log M(r_j,f)}{r_j^\rho} \geq \limsup_{j\to\infty} \frac{1}{\rho}\frac{\beta}{e}\frac{(r_j)^\rho}{(r_j)^\rho} = \frac{\beta}{\rho e}.$$

Since $\beta < \nu$ is arbitrary, this implies $\tau \geq \nu/\rho e$. $\square$