8. Phragmén-Lindelöf theorems

Theorem 8.1. *Suppose $f(z)$ is analytic inside a sectorial domain of opening π/α, where $\alpha > 1$, see the adjacent figure. Moreover, assume that $f(z)$ is continuous on the closure of the sectorial domain. If $|f(z)| \le M$ on the boundary of the domain and*

$$|f(z)| \le K e^{|z|^\beta}$$

inside of the domain for some constant $\beta < \alpha$, then $|f(z)| \le M$ inside of the domain.

Proof. By a rotation, we may assume that the domain in question is $\{ z \neq 0 \mid |\arg z| < \pi/2\alpha \}$. Choose now $\varepsilon > 0$ and γ such that $\beta < \gamma < \alpha$, and consider

$$F(z) := e^{-\varepsilon z^\gamma} f(z),$$

where $z^\gamma = (re^{i\varphi})^\gamma = r^\gamma e^{i\gamma\varphi}$.

Since

$$\mathrm{Re}(z^\gamma) = \mathrm{Re}(r^\gamma e^{i\gamma\varphi}) = \mathrm{Re}\Big(r^\gamma \big(\cos(\gamma\varphi) + i\sin(\gamma\varphi)\big)\Big) = r^\gamma \cos(\gamma\varphi),$$

we observe that

$$|F(z)| = |F(re^{i\varphi}) = e^{\mathrm{Re}(-\varepsilon z^\gamma)}|f(z)| = e^{-\varepsilon r^\gamma \cos(\gamma\varphi)}|f(z)|.$$

Since $|\gamma\varphi| < \frac{\gamma\pi}{2\alpha} < \frac{\pi}{2}$ for the closed sectorial domain, $\cos(\gamma\varphi) > 0$ and so $\exp\big(-\varepsilon r^\gamma \cos(\gamma\varphi)\big) < 1$, hence

$$|F(z)| \le |f(z)|$$

in the closed domain. In particular, $|F(z)| \le M$ on the boundary of the domain. In the open sector,

$$|F(re^{i\varphi})| = e^{-\varepsilon r^\gamma \cos(\gamma\varphi)}|f(z)| \le K e^{r^\beta - \varepsilon r^\gamma \cos(\gamma\varphi)}.$$

Since $\gamma > \beta$, $r^\beta - \varepsilon r^\gamma \cos(\gamma\varphi) \to -\infty$ as $r \to +\infty$; therefore

$$|F(re^{i\varphi})| \le M$$

for r large enough. Therefore, by the maximum principle, applied for the shaded domain in the adjacent figure, $|F(z)| \le M$ in the whole shaded domain. Since r may be taken arbitrarily large, the inequality $|F(z)| \le M$ in the whole open sector. Therefore,

$$|f(z)| \le M e^{\varepsilon r^\gamma \cos(\gamma\varphi)} \le M e^{\varepsilon r^\gamma}.$$

Letting $\varepsilon \to 0$, we get the assertion. $\square$

Theorem 8.2. *Change the estimate for f in the open sector to*

$$|f(z)| \leq Ke^{\delta|z|^\alpha} = K(\delta)e^{\delta|z|^\alpha}$$

for every $\delta > 0$, and keep the remaining assumptions unchanged. Then the same conclusion holds.

Proof. Again, we may assume the sector to be $|\varphi| \leq \frac{\pi}{2\alpha}$. Given $\varepsilon > 0$, define

$$F(z) := e^{-\varepsilon z^\alpha} f(z).$$

If $\delta < \varepsilon$, then we get on the real axis

$$|f(x)| \leq Ke^{\delta x^\alpha}$$

and

$$|F(x)| \leq Ke^{-\varepsilon x^\alpha} e^{\delta x^\alpha} = Ke^{(\delta-\varepsilon)x^\alpha} \to 0 \qquad \text{as } x \to \infty.$$

Since $|F(x)| \geq 0$ is continuous, we get, for a finite M',

$$|F(x)| \leq M' := \max\{\, |F(t)| \mid t \geq 0 \,\}$$

for all $x \geq 0$. Consider now $F(z)$ in the upper and lower half-sectors. Defining $M'' := \max(M, M')$, we see that the inequality $|F(z)| \leq M''$ holds on the boundaries of both half-sectors and $|F(z)| \leq Ke^{\delta r^\alpha}$ inside of the half-sectors. For φ such that $|\varphi| \leq \frac{\pi}{2\alpha}$, obviously

$$e^{-\varepsilon r^\alpha \cos(\varphi\alpha)} \leq e^{+\varepsilon r^\alpha}$$

and so, for some $K' > 0$,

$$|F(z)| = |e^{-\varepsilon z^\alpha}||f(z)| \leq Ke^{-\varepsilon r^\alpha \cos(\varphi\alpha)} e^{\delta r^\alpha} \leq Ke^{(\delta+\varepsilon)r^\alpha} \leq K'e^{r^\beta}$$

for any β such that $\alpha < \beta < 2\alpha$. By Theorem 8.1, $|F(z)| \leq M''$ in both half-sectors, and therefore in the whole sector $|\varphi| \leq \frac{\pi}{2\alpha}$.

Assume now that $M' > M$, hence $M'' = M' > M$. Since $F(x) \to 0$ as $x \to \infty$ and $|F(0)| \leq M$, there must exist a point $x_0 \in (0, +\infty)$ such that $|F(x_0)| = M' = M''$. By the maximum principle, F must be identically equal to the constant M', a contradiction. Therefore, we must have $M' \leq M$ and so $M'' = M$. This implies that $|F(z)| \leq M$ in the whole sector. But this means that

$$|f(z)| \leq M|e^{\varepsilon z^\alpha}|.$$

Letting now $\varepsilon \to 0$, the assertion follows. □

Theorem 8.3. *Suppose $f(z) \to a$ as $z \to \infty$ along two half-lines starting from the origin, and assume that $f(z)$ is analytic and bounded in one of the sectors between these two half-lines. Then $f(z) \to a$ uniformly as $r \to \infty$ in that sector.*

Proof. Considering $f(z) - a$, if needed, we may assume that $a = 0$. Moreover, if needed, we may consider $g(\zeta) = f((\zeta)^2)$ to achieve that the sector to be treated is $< \pi$. Finally, we may restrict us to considering the case of two half-lines $\pm\varphi$, $\varphi < \frac{\pi}{2}$, by an additional rotation.

Take now an arbitrary $\varepsilon > 0$. Clearly, we may assume that $|f(z)| \le M$ in the closed sector, while on the boundary half-lines, $|f(z)| < \varepsilon$ for all $r > r_1 = r_1(\varepsilon)$. Denote now $\lambda = \frac{r_1 M}{\varepsilon} > 0$ and define

$$F(z) = \frac{z}{z+\lambda} f(z).$$

Then

$$|F(z)| = \frac{r}{(r^2 + 2\lambda \operatorname{Re} z + \lambda^2)^{1/2}} |f(z)| < \frac{r}{(r^2+\lambda^2)^{1/2}} |f(z)|.$$

Now, for $r \le r_1$,

$$|F(z)| < \frac{r|f(z)|}{(r^2+\lambda^2)^{1/2}} \le \frac{rM}{\lambda} \le \frac{r_1 M}{\lambda} = \varepsilon$$

and on the boundary half-lines

$$|F(z)| < |f(z)| < \varepsilon,$$

provided $r > r_1$. Inside of the open sector, uniformly as $r \to \infty$,

$$|F(z)| < |f(z)| \le M \le Me^r \le Me^{r^\beta} \le Me^{r^\alpha}$$

for any α, β such that $1 < \beta < \alpha$. Since the opening of the sector is $< \pi$, we may take some $\alpha > 1$ such that the opening equals to $\frac{\pi}{\alpha}$. By Theorem 8.1, $|F(z)| \le \varepsilon$ in the closed sector. Therefore,

$$|f(z)| = \left|1 + \frac{\lambda}{z}\right| |f(z)| \le \left(1 + \frac{\lambda}{r}\right) |F(z)| \le 2\varepsilon$$

for all $r > \lambda$. Since $\varepsilon > 0$ is arbitrary, $f(z) \to 0$ uniformly as $r \to \infty$ inside of the sector. □

Theorem 8.4. *Suppose $f(z) \to a$ along a half-line starting from the origin and $f(z) \to b$ along a second half-line, again starting from the origin. Moreover, suppose that f is analytic and bounded in one of the two sectors between these half-lines. Then $a = b$ and $f(z) \to a$ uniformly in that sector as $r \to \infty$.*

Proof. Suppose that $f(z) \to a$ along $\varphi = \alpha$ and $f(z) \to b$ along $\varphi = \beta$, and that $\alpha < \beta$. Consider now, instead of f, the function

$$g(z) := \left(f(z) - \frac{a+b}{2}\right)^2.$$

It is now immediate to observe that

$$g(z) \to \left(a - \frac{a+b}{2}\right)^2 = \tfrac{1}{4}(a-b)^2$$

on $\varphi = \alpha$ and

$$g(z) \to \left(b - \frac{a+b}{2}\right)^2 = \tfrac{1}{4}(a-b)^2.$$

By Theorem 8.3, $g(z) \to \frac{1}{4}(a-b)^2$ uniformly in the sector as $r \to \infty$. Therefore,

$$g(z) - \tfrac{1}{4}(a-b)^2 = \big(f(z) - \tfrac{1}{2}(a+b)\big)^2 - \tfrac{1}{4}(a-b)^2 = \big(f(z)-a\big)\big(f(z)-b\big) \to 0$$

in the whole sector, uniformly as $r \to \infty$. Take now a circular arc, centred at the origin, such that

$$|f(z)-a||f(z)-b| \le \varepsilon$$

along this arc, inside of the closed sector. Then, at every point of this arc,

$$|f(z)-a| \le \sqrt{\varepsilon} \quad \text{or} \quad |f(z)-b| \le \sqrt{\varepsilon}.$$

If one of these inequalities holds on the whole arc, say $|f(z)-a| \le \sqrt{\varepsilon}$, and assuming that this circular arc has a radius large enough, then at the endpoint with $\varphi = \beta$, we get

$$|a-b| \le |f(z)-a| + |f(z)-b| \le 2\sqrt{\varepsilon}.$$

If this is not the case, then denote the two non-empty parts of the arc as $\Gamma_a = \{\, z \mid |f(z)-a| \le \sqrt{\varepsilon}\,\}$ and $\Gamma_b = \{\, z \mid |f(z)-b| \le \sqrt{\varepsilon}\,\}$. These are now closed sets and their union clearly equals to the whole circular arc. If their intersection would be empty, then, by elementary topology, one of these sets had to be empty, reducing to the previous case. Therefore, we may take a point z_0 from the intersection. Then

$$|a-b| \le |f(z_0)-a| + |f(z_0)-b| \le 2\sqrt{\varepsilon}.$$

Letting now $\varepsilon \to 0$, we get $a = b$. By Theorem 8.3, we get the assertion. □

Remark. Several variants of the Phragmén-Lindelöf theorems can be found in the literature, including also various regions, instead of sectors only.