9. Zeros of entire functions

Let $f(z)$ be an entire function and consider a disk $|z| \le r$ centred at $z = 0$. If r is large enough and $f(z)$ is a polynomial of degree n, then $f(z) = \alpha$ has n roots in $|z| \le r$. Moreover $M(r, f) \sim r^n$ on the boundary of the disk. This connection between the number of a-points and the maximum modulus carries over to transcendental entire functions. This is a deep property; moreover, some exceptional values α may appear.

Definition 9.1. Let (r_j) be a sequence of real numbers such that $0 < r_1 \le r_2 \le \cdots$. The *convergence exponent* λ for (r_j) will be defined by setting

$$\lambda = \inf\Big\{\ \alpha > 0 \ \Big|\ \sum_{j=1}^{\infty} (r_j)^{-\alpha} \text{ converges } \Big\}.$$

Remark. If $\sum_{j=1}^{\infty} r_j^{-\alpha}$ diverges for all $\alpha > 0$, then $\lambda = +\infty$ as the infimum of an empty set.

Definition 9.2. Let $f(z)$ be entire and let (z_n) be the zero-sequence of $f(z)$, deleting the possible zero at $z = 0$, every zero $\ne 0$ repeated according to its multiplicity, and arranged according to increasing moduli, i.e. $0 < |z_1| \le |z_2| \le \cdots$. The convergence exponent $\lambda(f)$ (for the zero-sequence of f) is now

$$\lambda(f) := \inf\Big\{\ \alpha > 0 \ \Big|\ \sum_{j=1}^{\infty} |z_j|^{-\alpha} \text{ converges } \Big\}.$$

Definition 9.3. Denote by $n(t) = n(t, \frac{1}{f})$ the number of zeros of $f(z)$ in $|z| \le t$, each zero counted according to its multiplicity.

Remark. In what follows, we assume that $f(0) \ne 0$. This is no essential restriction, since we may always replace $n(t)$ by $n(t) - n(0)$ below, if $f(0) = 0$.

Lemma 9.4. *The series* $\sum_{j=1}^{\infty} |z_j|^{-\alpha}$ *converges if and only if* $\int_0^{\infty} n(t) t^{-(\alpha+1)}\, dt$ *converges.*

Proof. Observe that $n(t)$ is a step function: zeros of $f(z)$ are situated on countably many circles centred at $z = 0$. Between these radii, $n(t)$ is constant and so $dn(t) = 0$ for these intervals. Passing over these radii $dn(t)$ jumps by an integer equals to the number of zeros on the circle. Therefore,

$$\sum_{j=1}^{N} |z_j|^{-\alpha} = \int_0^T \frac{dn(t)}{t^\alpha}, \qquad \text{where } T = |z_N|.$$

By partial integration,

$$\int_0^T \frac{dn(t)}{t^\alpha} = \Big/_{\!0}^{T} \frac{n(t)}{t^\alpha} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\, dt = \frac{n(T)}{T^\alpha} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\, dt.$$

Assume now that $\sum_{j=1}^{\infty} |z_j|^{\alpha}$ converges. Then, for each T,

$$\alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\, dt \le \int_0^T \frac{dn(t)}{t^\alpha} = \sum_{j=1}^{N} |z_j|^{-\alpha} \le \sum_{j=1}^{\infty} |z_j|^{-\alpha} < +\infty.$$

Therefore, $\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt$ converges.

Conversely, assume that the integral converges. Then

$$\frac{n(T)}{T^\alpha}(1-2^{-\alpha})\frac{1}{\alpha} = n(T)\int_T^{2T}\frac{dt}{t^{\alpha+1}} \le \int_T^{2T}\frac{n(t)}{t^{\alpha+1}}\,dt \le \int_0^\infty \frac{n(t)\,dt}{t^{\alpha+1}} =: K < +\infty.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^N |z_j|^{-\alpha} &= \frac{n(T)}{T^\alpha} + \alpha\int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt \\ &\le \frac{K\alpha}{1-2^{-\alpha}} + \alpha\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt = \frac{K\alpha}{1-2^{-\alpha}} + \alpha K < +\infty\end{aligned}$$

for each N. Therefore, $\sum_{j=1}^\infty |z_j|^{-\alpha}$ converges. □

Corollary 9.5. *Let $f(z)$ be an entire function, $f(0) \neq 0$. Then*

$$\lambda(f) = \inf\Big\{\, \alpha > 0 \;\Big|\; \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges} \Big\}.$$

Theorem 9.6. $\lambda(f) = \limsup_{r\to\infty} \dfrac{\log n(r)}{\log r}$.

Proof. Denote

$$\sigma := \limsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

Given $\varepsilon > 0$, there exists r_ε such that

$$n(r) \le r^{\sigma+\varepsilon}$$

for all $r \ge r_\varepsilon$. Then

$$\begin{aligned}\int_0^M \frac{n(t)}{t^{\alpha+1}}\,dt &= \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^M \frac{n(t)\,dt}{t^{\alpha+1}} \\ &\le \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^M t^{\sigma-\alpha-1+\varepsilon}\,dt.\end{aligned}$$

As $M \to \infty$, this converges, if $\sigma - \alpha - 1 + \varepsilon < -1 \implies \alpha > \sigma + \varepsilon$. Now, this is true for *all* $\alpha > 0$ such that $\alpha > \sigma + \varepsilon$. Therefore

$$\inf\Big\{\, \alpha > 0 \;\Big|\; \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges } \Big\} \le \sigma + \varepsilon.$$

By Corollary 9.5, $\lambda(f) \le \sigma + \varepsilon$ and so $\lambda(f) \le \sigma$.

To prove the converse inequality, we may assume that $\sigma > 0$. Take $\varepsilon > 0$ such that $\varepsilon < \sigma$. Then there is a sequence $r_j \to +\infty$ such that

$$\frac{\log n(r_j)}{\log r_j} \ge \sigma - \varepsilon,$$

hence

$$n(r_j) \geq r_j^{\sigma-\varepsilon}.$$

Take now any $\alpha > 0$ such that $0 < \alpha < \sigma - \varepsilon$. For each j, select

$$s_j \geq 2^{1/\alpha} r_j.$$

Since $n(t)$ is increasing, we get

$$\begin{aligned}\int_{r_j}^{s_j} \frac{n(t)\,dt}{t^{\alpha+1}} &\geq n(r_j) \int_{r_j}^{s_j} \frac{dt}{t^{\alpha+1}} \geq r_j^{\sigma-\varepsilon} \frac{1}{\alpha} \left(\frac{1}{r_j^\alpha} - \frac{1}{s_j^\alpha} \right) \\ &\geq \frac{1}{\alpha} r_j^{\sigma-\varepsilon} \frac{1}{r_j^\alpha} (1 - \tfrac{1}{2}) = \frac{1}{2\alpha} r_j^{\sigma-\alpha-\varepsilon}.\end{aligned}$$

Since $\alpha < \sigma - \varepsilon$, and so $\sigma - \alpha - \varepsilon > 0$, we see that

$$\int_{r_j}^{s_j} \frac{n(t)}{t^{\alpha+1}}\,dt \to +\infty \qquad \text{as } j \to \infty.$$

Therefore, $\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt$ diverges for all α, $0 < \alpha < \sigma - \varepsilon$. This means that

$$\inf \left\{ \alpha > 0 \;\middle|\; \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges} \right\} \geq \sigma - \varepsilon.$$

Therefore $\lambda(f) \geq \sigma - \varepsilon \implies \lambda(f) \geq \sigma$. $\square$

Theorem 9.7. (Jensen). *Let $f(z)$ be entire such that $f(0) \neq 0$ and denote*

$$N(r) = N\left(r, \frac{1}{f}\right) = \int_o^r \frac{n(t)}{t}\,dt.$$

Assume that there are no zeros of f on the circle $|z| = r > 0$. Then

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\varphi})|\,d\varphi - \log|f(0)|.$$

Remark. The restriction for zeros on $|z| = r$ is unessential, and may be removed by a rather complicated reasoning.

Proof. Let $a_1, a_2, \ldots, a_n$ be the zeros of f in $|z| \leq r$. Consider

$$g(z) := f(z) \prod_{j=1}^n \frac{r^2 - \overline{a}_j z}{r(z - a_j)}.$$

Then $g(z) \neq 0$ in $|z| \leq R$ for an $R > r$. For $|z| < \rho < R$, $\rho \neq r$, this is clear. If $|z| = r$, we see that ($z = re^{i\varphi}$)

$$\left| \frac{r^2 - \overline{a}_j z}{r(z - a_j)} \right| = \left| \frac{r^2 - \overline{a}_j r e^{i\varphi}}{r^2 e^{i\varphi} - a_j r} \right| = \left| \frac{r - \overline{a}_j e^{i\varphi}}{r - a_j e^{-i\varphi}} \right| = \left| \frac{r - \overline{a_j e^{-i\varphi}}}{r - a_j e^{-i\varphi}} \right| = 1$$

and so $|g(z)| = |f(z)| \neq 0$. Since $g \neq 0$ in $|z| < R$, it is an elementary computation (by making use of Cauchy–Riemann equations) that $\log|g(z)|$ is harmonic in $|z| < R$, i.e. that $\Delta\big(\log|g(z)|\big) \equiv 0$. By the mean value property of harmonic functions, CAI, Theorem 10.5, that

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\varphi})|\,d\varphi.$$

Since

$$|g(0)| = |f(0)|\prod_{j=1}^{n}\frac{r}{|a_j|},$$

we get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\varphi})|\,d\varphi &= \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\varphi})|\,d\varphi \\ &= \log|g(0)| = \log\Big(|f(0)|\prod_{j=1}^{n}\frac{r}{|a_j|}\Big) = \log|f(0)| + \sum_{j=1}^{n}\log\frac{r}{|a_j|}.\end{aligned}$$

Comparing this to the assertion, we observe that

$$\int_0^r \frac{n(t)}{t}\,dt = \sum_{j=1}^{n}\log\frac{r}{|a_j|}$$

remains to be proved. Denote $r_j = |a_j|$. Then

$$\begin{aligned}\sum_{j=1}^{n}\log\frac{r}{|a_j|} &= \sum_{j=1}^{n}\log\frac{r}{r_j} = \log\Big(\prod_{j=1}^{n}\log\frac{r}{r_j}\Big) = \log\frac{r^n}{r_1\cdots r_n} \\ &= n\log r - \sum_{j=1}^{n}\log r_j = \sum_{j=1}^{n-1} j(\log r_{j+1} - \log r_j) + n(\log r - \log r_n) \\ &= \sum_{j=1}^{n-1} j\int_{r_j}^{r_{j+1}}\frac{dt}{t} + n\int_{r_n}^{r}\frac{dt}{t} = \int_0^r\frac{n(t)}{t}\,dt. \quad \Box\end{aligned}$$

Remark. Given $\varphi\colon [r_0, +\infty) \to (0, +\infty)$, the Landau symbols $O\big(\varphi(r)\big)$ and $o\big(\varphi(r)\big)$ are frequently used. They mean any quantity $f(r)$ such that

For $O\big(\varphi(r)\big)$: $\exists K > 0$ such that $|f(r)/\varphi(r)| \leq K$ for r sufficiently large,

for $o\big(\varphi(r)\big)$: $\lim_{r\to\infty}\frac{f(r)}{\varphi(r)} = 0$.

Theorem 9.8. *Let $f(z)$ be entire of order ρ. Then for each $\varepsilon > 0$, $n(r) = O(r^{\rho+\varepsilon})$.*

Proof. We may assume that $|f(0)| \geq 1$ by multiplying f by a constant, if needed. By the Jensen formula

$$N(r) \leq \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\varphi})|\,d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi}\log M(r,f)\,d\varphi = \log M(r,f).$$

By the order, $\log M(r,f) \le r^{\rho+\varepsilon}$ for all r sufficiently large. Since $n(t)$ is increasing,

$$n(r)\log 2 = n(r)\int_r^{2r} \frac{dt}{t} \le \int_r^{2r} \frac{n(t)\,dt}{t} \le \int_0^{2r} \frac{n(t)\,dt}{t}$$
$$= N(2r) \le \log M(2r,f) \le (2r)^{\rho+\varepsilon} = 2^{\rho+\varepsilon} r^{\rho+\varepsilon}$$

for r sufficiently large. Therefore

$$n(r) \le \left(\frac{1}{\log 2}\cdot 2^{\rho+\varepsilon}\right) r^{\rho+\varepsilon}. \quad \square$$

Theorem 9.9. *For any entire function $f(z)$, $\lambda(f) \le \rho(f)$.*

Proof. By Theorem 9.8, given $\varepsilon > 0$, there exists $K > 0$ such that

$$n(r) \le K r^{\rho+\varepsilon}, \qquad \rho = \rho(f)$$

for r sufficiently large, say $r \ge r_0$. Then

$$\int_0^M \frac{n(t)}{t^{\alpha+1}}\,dt = \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\,dt + \int_{r_0}^M \frac{n(t)\,dt}{t^{\alpha+1}} \le \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\,dt + K\int_{r_0}^M t^{\rho+\varepsilon-\alpha-1}\,dt$$

If now $\alpha > \rho+\varepsilon$, then $\rho+\varepsilon-\alpha-1 < -1$, and therefore the last integral converges as $M \to \infty$, hence

$$\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \quad \text{converges.}$$

This means that $\lambda(f) \le \rho+\varepsilon$ and so $\lambda(f) \le \rho(f)$. $\square$