Key to Demonstration 2

I.

The equality case is to say $\forall\alpha,\beta\in\mathbb{C}$, $|\alpha+\beta|=|\alpha|+|\beta|$ if and only if $\alpha=t\beta$ for some $t\geq 0$.

Proof.

$\Leftarrow$ If $\alpha=t\beta$, $t>0$, then $|\alpha+\beta|=|t\beta+\beta|=(t+1)|\beta|=t|\beta|+|\beta|=|t\beta|+|\beta|=|\alpha|+|\beta|$.

$\Rightarrow$ Since $|\alpha+\beta|=|\alpha|+|\beta|$, $|\alpha+\beta|^2=(|\alpha|+|\beta|)^2$, then $\mathrm{Re}(\bar{\alpha}\beta)=|\alpha\beta|=|\bar{\alpha}\beta|$, that is to say $\bar{\alpha}\beta\in\mathbb{R}$, and $\alpha\beta>0$, then we have the following discussion:

- $\alpha,\beta\in\mathbb{R}$, it is easy to find out a $t>0$, such that $\alpha=t\beta$, $t>0$;
- if one of $\alpha,\beta\in\mathbb{C}$, then the other must also belong to $\mathbb{C}$, and $\alpha=t\beta$, where $t>0$.

Thus finishing the proof.

II.

$|\alpha+\beta|^2=(\alpha+\beta)(\bar{\alpha}+\bar{\beta})=|\alpha|^2+|\beta|^2+2\mathrm{Re}(\bar{\alpha}\beta)$, one the other side $\big||\alpha|-|\beta|\big|^2=(|\alpha|-|\beta|)^2=|\alpha|^2+|\beta|^2-2|\alpha||\beta|$. Since we have know that

$$-|\alpha||\beta|=-|\bar{\alpha}\beta|\leq\mathrm{Re}(\bar{\alpha}\beta)\leq|\bar{\alpha}\beta|=|\alpha\beta|=|\alpha||\beta|\,,$$

$|\alpha+\beta|\geq\big||\alpha|-|\beta|\big|$.

III.

(a)

$$\begin{aligned}a&=(1+i)^{1/2}=\Big[\sqrt{2}\Big(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\Big)\Big]^{1/2}=2^{1/4}\Big(\cos(\frac{\pi}{4}+2k\pi)+i\sin(\frac{\pi}{4}+2k\pi)\Big)^{1/2}\\&=2^{1/4}\Big(\cos(\frac{\pi}{8}+k\pi)+i\sin(\frac{\pi}{8}+k\pi)\Big),\end{aligned}$$

then we can get

$$\begin{aligned}a&=\cos(\frac{\pi}{8})+i\sin(\frac{\pi}{8}), && k=0\\a&=\cos(\frac{\pi}{8}+\pi)+i\sin(\frac{\pi}{8}+\pi)=-\cos(\frac{\pi}{8})-i\sin(\frac{\pi}{8}), && k=1\end{aligned}$$

(b)

$$b=(-16)^{3/4}=[16(\cos\pi+i\sin\pi)]^{3/4}=8\Big(\cos(\frac{3}{4}\pi+\frac{3k\pi}{2})+i\sin(\frac{3}{4}\pi+\frac{3k\pi}{2})\Big),$$

then we can get

$$\begin{aligned}b&=8\Big(\cos(\frac{3}{4}\pi)+i\sin(\frac{3}{4}\pi)\Big)=-4\sqrt{2}+4\sqrt{2}i, && k=0;\\b&=8\Big(\cos(\frac{3}{4}\pi+\frac{3}{2}\pi)+i\sin(\frac{3}{4}\pi+\frac{3}{2}\pi)\Big)=8(\cos\frac{9}{4}\pi+i\sin\frac{9}{4}\pi)=4\sqrt{2}+4\sqrt{2}i, && k=1;\\b&=8\Big(\cos(\frac{3}{4}\pi+3\pi)+i\sin(\frac{3}{4}\pi+3\pi)\Big)=8\Big(\cos(-\frac{1}{4}\pi)+i\sin(-\frac{1}{4}\pi)\Big)=4\sqrt{2}-4\sqrt{2}i, && k=2;\\b&=8\Big(\cos(\frac{3}{4}\pi+\frac{9}{2}\pi)+i\sin(\frac{3}{4}\pi+\frac{9}{2}\pi)\Big)=8\Big(\cos(\frac{5}{4}\pi)+i\sin(\frac{5}{4}\pi)\Big)=-4\sqrt{2}-4\sqrt{2}i; && k=3.\end{aligned}$$

(c)

$$\begin{aligned} c = (1+i)^{5/3} &= \Big[\sqrt{2}\Big(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\Big)\Big]^{5/3} \\ &= 2^{5/6}\Big(\cos(\frac{\pi}{4}+2k\pi)+i\sin(\frac{\pi}{4}+2k\pi)\Big)^{5/3} \\ &= 2^{5/6}\Big(\cos(\frac{5}{12}\pi+\frac{4}{3}k\pi)+i\sin(\frac{5}{12}\pi+\frac{4}{3}k\pi)\Big), \end{aligned}$$

then we can get

$$c = 2^{5/6}(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi), \qquad k=0$$
$$c = 2^{5/6}\Big(\cos(\frac{5}{12}\pi+\frac{4}{3}\pi)+i\sin(\frac{5}{12}\pi+\frac{4}{3}\pi)\Big) = 2^{5/6}\Big(\cos\frac{1}{4}\pi - i\sin\frac{1}{4}\pi\Big), \qquad k=1$$
$$c = 2^{5/6}\Big(\cos(\frac{5}{12}\pi+\frac{2}{3}\pi)+i\sin(\frac{5}{12}\pi+\frac{2}{3}\pi)\Big) = -2^{5/6}\Big(\cos\frac{1}{12}\pi + i\sin\frac{1}{12}\pi\Big), \qquad k=2.$$

IV.

(a) Since it is easy to see that $\lim_{n\to\infty}\frac{1}{n}=0$, and $\lim_{n\to\infty}\frac{n+2}{n}=1$,

$$\lim_{n\to\infty}\left(\frac{1}{n}+(\frac{n+2}{n})i\right) = \lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{n+2}{n}i = i.$$

(b) $\left(\frac{1}{\sqrt{3}}+\frac{i}{\sqrt{3}}\right)^n = \Big[\frac{\sqrt{2}}{\sqrt{3}}\Big(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\Big)\Big]^n = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^n E(\frac{n\pi}{4})$. Since for any $\epsilon>0$, there exists $N=\log_{\frac{\sqrt{2}}{\sqrt{3}}}\epsilon$, when $n>N$, $\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^n<\epsilon$, and $|E(\frac{n\pi}{4})|$ is bounded definitely, we get finally that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{3}}+\frac{i}{\sqrt{3}}\right)^n=0$.

(c) Since $\lim_{n\to\infty}\big(1+\frac{1}{n}\big)^n=e$, $\lim_{n\to\infty}\left(\big(1+\frac{1}{n}\big)^n+\big(1+\frac{1}{n}\big)^{-n}i\right)=e+\frac{i}{e}$.

V.

$\lim_{n\to\infty}z_n=\alpha$, $\forall\varepsilon>0$, $\exists N_1\in\mathbb{N}$, when $n\ge N_1$, $|z_n-\alpha|<\varepsilon$.
$\lim_{n\to\infty}\zeta_n=\beta$, $\forall\varepsilon>0$, $\exists N_2$, when $n\ge N_2$, $|\zeta_n-\beta|<\varepsilon$.
Then (a) $\forall\varepsilon>0$, choose $N=\max\{N_1,N_2\}$, when $n\ge N$,

$$|z_n+\zeta_n-(\alpha+\beta)| = |z_n-\alpha+(\zeta_n-\beta)| \le |z_n-\alpha|+|\zeta_n-\beta| < \varepsilon+\varepsilon=2\varepsilon,$$

so $\lim_{n\to\infty}(z_n+\zeta_n)=\alpha+\beta$. In the same way, we can get that $\lim_{n\to\infty}(z_n-\zeta_n)=\alpha-\beta$.

(b) Since $\lim_{n\to\infty}z_n=\alpha$, there exists a N_3, s.t. when $n\ge N_3$, there exists a constant $M>0$, and$|z_n|\le M$. $\forall\varepsilon>0$, choose $N=\max\{N_1,N_2,N_3\}$, when $n\ge N$,

$$\begin{aligned} |z_n\zeta_n-\alpha\beta| =& |z_n\zeta_n-z_n\beta+z_n\beta-\alpha\beta| \le |z_n\zeta_n-z_n\beta|+|z_n\beta-\alpha\beta| \\ \le& |z_n||\zeta_n-\beta|+|z_n-\alpha||\beta| \le M\varepsilon+\varepsilon|\beta| = (M+|\beta|)\varepsilon, \end{aligned}$$

We get $\lim_{n\to\infty}(z_n\zeta_n) = \alpha\beta$.

VI.

First, let's consider the series $\sum nr^n$, where $0 < r < 1$, by the convergence theorem of series, $R = \lim_{n\to\infty} \frac{(n+1)r^{n+1}}{nr^n} = r < 1$, so the series $\sum_{n=0}^{\infty} n|z^n| = \sum_{n=0}^{\infty} n|z|^n$ convergent in the unit disk. Thus $\lim_{n\to\infty} n|z|^n = 0$, and so does $\lim_{n\to\infty} nz^n$. In the same way, we can show that $\lim_{n\to\infty} n^2z^n = \lim_{n\to\infty} n(n+1)z^n = 0$.

VII.

(a) $\int_0^1 (2+ip^2)dp = \int_0^1 2dp + i\int_0^1 p^2dp = 2 + \frac{i}{3}$;

(b) $\int_{-2}^{5} f(s)ds = \int_{-2}^{2}(-1+7is^2)ds + \int_{2}^{5}(2s+is^2)ds = -4 + \frac{112}{3}i + 21 + \frac{117}{3}i = 17 + \frac{229}{3}i$.

VIII.

Since $|3+2y| \leq 3+2|y| = 3+2\times 3 = 9$, by Theorem 1.4 e), $\left|\int_C (3+2y)dz\right| \leq 9\times 6\pi = 54\pi$.