Key to Demonstration 1

I.
Key.

$$(a)3; \qquad (b)4;$$
$$(c)6; \qquad (d)-5.$$

II.
Key.

$$(\alpha\beta)\gamma = \big[(3+2i)(1-4i)\big]\left(\frac{1}{2}+3i\right) = [11-10i]\left(\frac{1}{2}+3i\right) = 35\frac{1}{2}+28i$$

and

$$\alpha(\beta\gamma) = (3+2i)\big[(1-4i)(\frac{1}{2}+3i)\big] = (3+2i)\left[12\frac{1}{2}+i\right] = 35\frac{1}{2}+28i.$$

III.
Key.

$$(a)\ 1/\alpha = \frac{1}{3+2i} = \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{13} = \frac{3}{13} - \frac{2}{13}i,$$
$$(b)\beta/\alpha = \frac{1-4i}{3+2i} = \frac{(1-4i)(3-2i)}{(3+2i)(3-2i)} = \frac{-5-14i}{13} = -\frac{5}{13} - \frac{14}{13}i.$$

IV.
Key.
$z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$, so

(a) Re $z^2 = x^2 - y^2$;

(b) Im $z^2 = 2xy$;

and

$$(1/z^2) = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2xyi} = \frac{x^2-y^2-2xyi}{(x^2-y^2+2xyi)(x^2-y^2-2xyi)}$$
$$= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2+4x^2y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}i,$$

so

(c) Re $(1/z^2) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$;

(d) Im $(1/z^2) = -\dfrac{2xy}{(x^2+y^2)^2}$.

V.

Key.

Let $\alpha = a + bi$, and $\beta = c + di$, if

$$(a + bi)(c + di) = ac - bd + (ad + bc)i = 0,$$

we know

$$\begin{cases} ac - bd = 0 \\ ad + bc = 0, \end{cases}$$

then we can get

$$\begin{cases} ac^2 - bdc = 0 & (1) \\ ad^2 + bcd = 0 & (2) \end{cases}$$

and

$$\begin{cases} acd - bd^2 = 0 & (3) \\ adc + bc^2 = 0 & (4). \end{cases}$$

Discussion: if $\beta \neq 0$, it means $cd \neq 0$, then (1)+(2)= $ac^2 + ad^2 = a(c^2 + d^2) = 0$, and (3)−(4)= $bd^2 + bc^2 = b(d^2 + c^2) = 0$, consequently, $a = 0, b = 0$, that is $\alpha = 0$. In the same way, if $\alpha \neq 0$, then we can get $\beta = 0$. Thus, $\alpha\beta = 0 \ \ (\alpha, \beta \in \mathbb{C})$ implies at least one of α and β is 0.

VI.

Key. Let $\alpha = a + bi$, and $\beta = c + di$,

(a) $\alpha + \bar{\alpha} = (a + bi) + (a - bi) = 2a = 2\text{Re } \alpha$;

(b) $\overline{\alpha + \beta} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = (a - bi) + (c - di) = \bar{\alpha} + \bar{\beta}$;

(c)

$$\overline{(\alpha/\beta)} = \overline{\frac{a + bi}{c + di}} = \overline{\frac{(a + bi)(c - di)}{(c + di)(c - di)}} = \overline{\frac{(ac + bd) - (ad - bc)i}{c^2 + d^2}} = \frac{ac + bd}{c^2 + d^2} + \frac{ad - bc}{c^2 + d^2}i$$

and

$$\bar{\alpha}/\bar{\beta} = \frac{a - bi}{c - di} = \frac{(a - bi)(c + di)}{(c - di)(c + di)} = \frac{(ac + bd) + (ad - bc)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{ad - bc}{c^2 + d^2}i,$$

thus $\overline{(\alpha/\beta)} = \bar{\alpha}/\bar{\beta}$;

(d) $|\alpha| = |a + bi| = \sqrt{a^2 + b^2}$ and $|\bar{\alpha}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$, so $|\alpha| = |\bar{\alpha}|$.

VII.

Key. First prove that for any complex number a and complex variable z, $\overline{az} = \bar{a}\bar{z}$. Let $a = \alpha + \beta i$ and $z = x + yi$, then

$$\overline{az} = \overline{(\alpha + \beta i)(x + yi)} = \overline{(\alpha x - \beta y) + (\alpha y + \beta x)i} = (\alpha x - \beta y) - (\alpha y + \beta x)i$$

and

$$\bar{a}\bar{z} = \overline{(\alpha + \beta i)(x + yi)} = (\alpha - \beta i)(x - yi) = (\alpha x - \beta y) - (\alpha y + \beta x)i,$$

so $\overline{az} = \bar{a}\bar{z}$.

Next, let's prove that $\overline{z^2} = \bar{z}^2$.

$$\overline{z^2} = \overline{(x+yi)^2} = \overline{x^2 - y^2 + 2xyi} = x^2 - y^2 - 2xyi$$

and

$$\bar{z}^2 = (x - yi)^2 = x^2 - y^2 - 2xyi,$$

so $\overline{z^2} = \bar{z}^2$.

Induction: when $n = 1$, it is easy to show that

$$\overline{P(z)} = \overline{a_0 + a_1 z} = \bar{a}_0 + \overline{a_1 z} = \bar{a}_0 + \bar{a}_1 \bar{z}.$$

Suppose when $n = k$, the result is true, that is

$$\overline{P(z)} = \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k} = \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k,$$

then when $n = k + 1$,

$$\begin{aligned}
\overline{P(z)} &= \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + a_{k+1} z^{k+1}} \\
&= \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k} + \overline{a_{k+1} z^{k+1}} \\
&= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k + \bar{a}_{k+1} \overline{z^{k+1}} \\
&= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k + \bar{a}_{k+1} \overline{z^k z} \\
&= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k + \bar{a}_{k+1} \overline{z^k}\, \bar{z} \\
&= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k + \bar{a}_{k+1} \bar{z}^k \bar{z} \\
&= \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}^2 \bar{z}^2 + \cdots + \bar{a}_k \bar{z}^k + \bar{a}_{k+1} \bar{z}^{k+1},
\end{aligned}$$

by induction, we can get the result.