Key to Demonstration 3

I. Proof. By Theorem 1.4 e), and the fact that $|z| = 1$, it is easy to get

$$\left|\int_C \frac{dz}{2z^2+5}\right| \le \max\left|\frac{1}{2z^2+5}\right|\left|\int_C dz\right| = \left|\frac{1}{2(-i)^2+5}\right| \cdot 2\pi = \frac{2\pi}{3}\,.$$

II. Key.

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}\left(\int_C \frac{dz}{z-2} - \int_C \frac{dz}{z+2}\right)\,.$$

(a)

$$\int_{|z|=1} \frac{dz}{z^2-4} = \frac{1}{4}(0-0) = 0\,;$$

(b)

$$\int_{|z|=4} \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i - 2\pi i) = 0\,;$$

(c)

$$\int_{|z-2|=2} \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i - 0) = \frac{\pi i}{2}\,.$$

III. Key.

$$\int_C \frac{dz}{z^2-1} = \frac{1}{2}\left(\int_C \frac{dz}{z-1} - \int_C \frac{dz}{z+1}\right)\,.$$

(a)

$$\int_{|z|=\frac{1}{3}} \frac{dz}{z^2-1} = \frac{1}{2}(0-0) = 0\,;$$

(b)

$$\int_{|z|=3} \frac{dz}{z^2-1} = \frac{1}{2}(2\pi i - 2\pi i) = 0\,;$$

(c)

$$\int_{|z-1|=1} \frac{dz}{z^2-1} = \frac{1}{2}(2\pi i - 0) = \pi i\,.$$

IV. Key.

$$\int_C \frac{dz}{z(z-1)(z+2)} = -\frac{1}{2}\int_C \frac{dz}{z} + \frac{1}{3}\int_C \frac{dz}{z-1} + \frac{1}{6}\int_C \frac{dz}{z-(-2)}\,.$$

(a)

$$\int\limits_{|z|=\frac{1}{2}} \frac{dz}{z(z-1)(z+2)} = -\frac{1}{2}\cdot 2\pi i + \frac{1}{3}\cdot 0 + \frac{1}{6}\cdot 0 = -\pi i\,;$$

(b)

$$\int\limits_{|z|=\frac{3}{2}} \frac{dz}{z(z-1)(z+2)} = -\frac{1}{2}\cdot 2\pi i + \frac{1}{3}\cdot 2\pi i + \frac{1}{6}\cdot 0 = -\frac{\pi}{3}i\,;$$

(c)

$$\int\limits_{|z|=\frac{5}{2}} \frac{dz}{z(z-1)(z+2)} = -\frac{1}{2}\cdot 2\pi i + \frac{1}{3}\cdot 2\pi i + \frac{1}{6}\cdot 2\pi i = 0\,;.$$

(d)

$$\int\limits_{|z-1|=\frac{1}{2}} \frac{dz}{z(z-1)(z+2)} = -\frac{1}{2}\cdot 0 + \frac{1}{3}\cdot 2\pi i + \frac{1}{6}\cdot 0 = \frac{2\pi}{3}i\,.$$

V. Key. Let $z = \cos\theta + i\sin\theta$, $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{i}{2}(\frac{1}{z} - z)$, then,

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{3\sin\theta + 5} &= \int\limits_{|z|=1} \frac{dz}{iz(\frac{3i}{2}(\frac{1}{z} - z) + 5)} \\ &= 2\int\limits_{|z|=1} \frac{dz}{(3z+i)(z+3i)} \\ &= \frac{-i}{4}\Big(\int\limits_{|z|=1} \frac{dz}{z+\frac{i}{3}} - \int\limits_{|z|=1} \frac{dz}{z+3i}\Big) \\ &= \frac{-i}{4}\cdot 2\pi i = \frac{\pi}{2}\,.\end{aligned}$$

VI. Key. Let $z = \cos\theta + i\sin\theta$, $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, then,

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int\limits_{|z|=1} \frac{dz}{iz(2+\frac{1}{2}(z+\frac{1}{z}))} &= \frac{2}{i}\int\limits_{|z|=1} \frac{dz}{z^2+4z+1} \\ =& \frac{2}{i}\int\limits_{|z|=1} \frac{dz}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))} \\ =& \frac{2}{i}\left[\frac{1}{2\sqrt{3}}\int\limits_{|z|=1} \frac{dz}{z-(-2+\sqrt{3})} - \frac{1}{2\sqrt{3}}\int\limits_{|z|=1} \frac{dz}{z-(-2-\sqrt{3})}\right] = \frac{2\pi}{\sqrt{3}}\,.\end{aligned}$$

VII. Proof. Let $z = \cos\theta + i\sin\theta$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, then

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \int_{|z|=1} \frac{dz/iz}{1 + \frac{1}{2}a(z + \frac{1}{z})} = \frac{2}{i} \int_{|z|=1} \frac{dz}{az^2 + 2z + a}$$
$$= \frac{2}{ia} \int_{|z|=1} \frac{dz}{\left(z + \frac{1+\sqrt{1-a^2}}{a}\right)\left(z + \frac{1-\sqrt{1-a^2}}{a}\right)}$$
$$= \frac{2}{ia} \frac{a}{2\sqrt{1-a^2}} \left[\int_{|z|=1} \frac{dz}{z + \frac{1-\sqrt{1-a^2}}{a}} - \int_{|z|=1} \frac{dz}{z + \frac{1+\sqrt{1-a^2}}{a}} \right]$$

Since $0 < a < 1$, we can get $-1 < \frac{-1+\sqrt{1-a^2}}{a} < 0$, because $\frac{-1+\sqrt{1-a^2}}{a} < 0$ and $\frac{-1+\sqrt{1-a^2}}{a} > -1 \;\Leftrightarrow\; 1-a < \sqrt{1-a^2} \;\Leftrightarrow\; (1-a^2)^2 < 1-a^2 \;\Leftrightarrow\; a^2-a < 0 \;\Leftrightarrow\; a(a-1) < 0 \;\Leftrightarrow\; a < 1$, thus we can get $\frac{-1+\sqrt{1-a^2}}{a} \in I(C)$ and $\frac{-1-\sqrt{1-a^2}}{a} \in E(C)$, where $C = \{z : \; |z| = 1\}$. So,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{1}{i\sqrt{1-a^2}}(2\pi i - 0) = \frac{2\pi}{\sqrt{1-a^2}}\,.$$