Key to Demonstration 4

I.
Solution

$$
\begin{aligned}
(a)\quad & \lim_{h\to 0} \frac{(z+h)^3+(z+h)^2-(z+h)+1-z^3-z^2+z-1}{h} \\
&= \lim_{h\to 0} \frac{3z^2h+3zh^2+h^3+2zh+h^2-h}{h} \\
&= \lim_{h\to 0} \frac{h(3z^2+3zh+h^2+2z+h-1)}{h} \\
&= \lim_{h\to 0}(3z^2+3zh+h^2+2z+h-1) = 3z^2+2z-1.
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad & \lim_{h\to 0} \frac{\frac{(z+h)^2-1}{(z+h)^2+1} - \frac{z^2-1}{z^2+1}}{h} \\
&= \lim_{h\to 0} \frac{[(z+h)^2-1](z^2)-(z^2)[(z+h)^2+1]}{h([(z+h)^2+1](z^2+1))} \\
&= \lim_{h\to 0} \frac{h(4z+2h)}{h([(z+h)^2+1](z^2+1))} \\
&= \lim_{h\to 0} \frac{(4z+2h)}{[(z+h)^2+1](z^2+1)} = \frac{4z}{(z^2+1)^2}.
\end{aligned}
$$

$$
\begin{aligned}
(c)\quad & \lim_{h\to 0} \frac{((z+h)^2-1)((z+h)^2-3(z+h))-(z^2-1)(z^2-3z)}{h} \\
&= \lim_{h\to 0} \frac{(z+h)^4-3(z+h)^3-(z+h)^2+3(z+h)-(z^4-3z^3-z^2+3z)}{h} \\
&= \lim_{h\to 0} \frac{h(4z^3+6z^2h+4zh^2+h^3-3h^2-9z^2-9zh-2z-h+3)}{h} \\
&= \lim_{h\to 0} 4z^3+6z^2h+4zh^2+h^3-3h^2-9z^2-9zh-2z-h+3 \\
&= 4z^3-9z^2-2z+3.
\end{aligned}
$$

II.
Solution For z near z_0,

$$|f(z)-f(z_0)| = \left|\frac{f(z)-f(z_0)}{z-z_0}\right||z-z_0| \to |f'(z_0)|\cdot 0 = 0$$

as $z \to z_0$. Hence, $\lim_{z\to z_0} f(z) = f(z_0)$. In particular, both the real and the imaginary parts of f are continuous on any domain on which f is analytic.

III.

Solution Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be any polynomial and $a_0, \cdots, a_n \in \mathbb{C}$. To show that $P(z)$ is analytic everywhere, we only need to show that $f(z) = z^n$, $n = 1, 2, \cdots$, is analytic everywhere.

$$\begin{aligned}(z+h)^n - z^n =& nz^{n-1}h + \frac{n(n-1)}{2} z^{n-2}h^2 + \cdots + h^n \\ =& h(nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}h + \cdots + h^{n-1}),\end{aligned}$$

so

$$\frac{(z+h)^n - z^n}{h} \to nz^{n-1} \quad \text{as} \quad h \to 0.$$

IV.

Solution We know that

$$e^{z+h} - e^z = e^z(e^h - 1),$$

by the properties of the exponential function. Furthermore, with $h = \sigma + i\tau$,

$$\begin{aligned}e^h - 1 - h =& [e^\sigma \cos\tau - 1 - \sigma] + i[e^\sigma \sin\tau - \tau] \\ =& [e^\sigma(\cos\tau - 1) + e^\sigma - 1 - \sigma] + i[e^\sigma(\sin\tau - \tau) + \tau(e^\sigma - 1)].\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{e^h - 1}{h} - 1\right| =& \left|\frac{e^h - 1 - h}{h}\right| \\ \leq& e^\sigma\left|\frac{1 - \cos\tau}{\tau}\right| + \left|\frac{e^\sigma - 1 - \sigma}{\sigma}\right| + e^\sigma\left|\frac{\sin\tau - \tau}{\tau}\right| + |e^\sigma - 1|.\end{aligned}$$

To obtain this inequality, we used the triangle inequality, as well as the simple facts that

$$\frac{1}{|h|} \leq \frac{1}{|\tau|} \quad \text{and} \quad \frac{1}{|h|} \leq \frac{1}{|\sigma|}.$$

However, each of the four quantities within absolute value signs approaches zero as σ and τ independently approach zero (use l'Hôpital's Rule on each, if you like), so

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

This finally gives

$$\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h\to 0} \frac{e^h - 1}{h} = e^z.$$

Consequently, e^z is differentiable at all points z, and $(e^z)' = e^z$.

V.

Solution Since $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If $u(x, y)$ is a constant, it is easy to get that $v(x, y)$ is a constant, so $f(z)$ is a constant, of course $|f(z)|$ is a constant.

If $|f(z)|$ is a constant, $u^2(x,y)+v^2(x,y)=C.$, thus we get

$$\begin{cases} u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial x}=0 \\ u\dfrac{\partial u}{\partial y}+v\dfrac{\partial v}{\partial y}=0, \end{cases}$$

with the Cauchy-Riemann equations, we can get

$$\begin{cases} u\dfrac{\partial u}{\partial x}-v\dfrac{\partial u}{\partial y}=0 \\ u\dfrac{\partial u}{\partial y}+v\dfrac{\partial u}{\partial x}=0, \end{cases}$$

so $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y}=0$, that is to say u is a constant, furthermore, $f(z)$ is a constant.

VI.

Solution (a) Let $h\to 0$ in x-axis direction, that is to say $h\in\mathbb{R}$, and $h\to 0$, then

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{x+h-iy-(x-iy)}{h}=1,$$

next, let $h\to 0$ in y-axis direction, that is to say $h=i\tau$, and $\tau\to 0$, then

$$\lim_{\tau\to 0}\frac{f(z+i\tau)-f(z)}{i\tau}=\lim_{\tau\to 0}\frac{x-i(y+\tau)-(x-iy)}{i\tau}=-1.$$

We can get different values when we take the limit in different ways, so $f(z)=\bar{z}$ is nowhere differentiable.

(b) With the same way showed as above,

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{x+h-x}{h}=1$$

$$\lim_{\tau\to 0}\frac{f(z+i\tau)-f(z)}{i\tau}=\lim_{\tau\to 0}\frac{x-x}{i\tau}=0,$$

we know that $f(z)=x$ is nowhere differentiable.