Key to Demonstration 5

I.

Solution $u = u + iv$, by Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, we have

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -(-6xy) = 6xy.$$

We know that $dv = \dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy$, so for any point (x_0, y_0), let $C = C_1 + C_2$ be a segmented line, $(0,0) \xrightarrow{C_1} (x_0, 0) \xrightarrow{C_2} (x_0, y_0)$,

$$\begin{aligned}
v(x_0, y_0) - v(0,0) = \int_C dv &= \int_C \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= \Big(\int_{C_1} + \int_{C_2}\Big)\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= \int_{C_1} 6xydx + \int_{C_2} (3x^2 - 3y^2)dy \\
&= 3\int_{C_2} (x_0^2 - y^2)dy = 3\int_0^{y_0} (x_0^2 - y^2)dy \\
&= 3\int_0^{y_0} \Big(x_0^2 y - \frac{y^3}{3}\Big)dy = 3(x_0^2 y_0 - \frac{y_0^3}{3}).
\end{aligned}$$

Thus $v(x_0, y_0) = 3x_0^2 y_0 - y_0^3 + v(0,0)$, then

$f(z) = f(x,y) = u(x,y) + iv(x,y) = x^3 - 3xy^2 + i(3x^2y - y^3) + iv(0,0) = (x+iy)^3 + iv(0,0) = z^3 + iv(0,0).$

II.

Solution Since $f(z) = \dfrac{1+z^2}{z^2-1} = \dfrac{1+z^2}{(z+1)(z-1)}$, it is easy to see that there exists N, s.t. $f(z)$ is analytic in $B(\infty, N) = \{z \in \mathbb{C}, \ \ |z| > N\}$.

Let $z = \frac{1}{\zeta}$, then

$$f(z) = f\left(\frac{1}{\zeta}\right) = \frac{1 + \frac{1}{\zeta^2}}{\frac{1}{\zeta} - 1} = \frac{\zeta^2 + 1}{1 - \zeta^2}$$

is analytic at 0, so $f(z)$ is analytic at ∞.

III.

Solution

$$(a) \quad \lim_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e} < 1,$$

so $\sum \alpha_n$ is convergent.

$$(b)\quad \lim_{n\to\infty} \sqrt[n]{\frac{n^3(n+1)^n}{(3n)^n}} = \lim_{n\to\infty} \sqrt[n]{n^3}\cdot\frac{n+1}{3n} = \lim_{n\to\infty} (\sqrt[n]{n})^3\Big(\frac{1}{3}+\frac{1}{3n}\Big) = \frac{1}{3} < 1,$$

so $\sum \alpha_n$ is convergent.

IV.

Solution $a = \lim \dfrac{(n+1)^2}{n^2} = \lim \left(1+\dfrac{1}{n}\right)^2 = 1$, and so $\rho = 1/a = 1$.

V.

Solution $a = \lim \sqrt[n]{[(n+1)/n]^{n^2}} = \lim \left(1+\dfrac{1}{n}\right)^n = e$, where $e \approx 2.71828\ldots$ is the base of natural logarithms. Hence $\rho = \frac{1}{e} \approx 0.367\ldots$.

VI.

Solution $a = \lim \dfrac{1}{\left(\frac{n+1}{n}\right)^p} = \lim \dfrac{1}{\left(1+\frac{1}{n}\right)^p} = \lim \dfrac{1}{e^{\frac{p}{n}}} = 1$, and so $\rho = 1/a = 1$.

$a = \lim \left(\dfrac{n+1}{n}\right)^p = \lim \left(1+\dfrac{1}{n}\right)^p = \lim e^{\frac{p}{n}} = 1$, and so $\rho = 1/a = 1$.

Supplement To show that $n^{\frac{1}{n}} \to 1$, we only need to see that $\log n^{\frac{1}{n}} = \dfrac{\log n}{n} \to 0$.

Similarly, to show that $a^{\frac{1}{n}} \to 1$, we only need to see that $\log a^{\frac{1}{n}} = \dfrac{\log a}{n} \to 0$.