Key to Demonstration 6

I.

Solution First let's show that $e^{iz} = \cos z + i \sin z$. We know that

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n:\ \text{even}} \frac{(iz)^n}{n!} + \sum_{n:\ \text{odd}} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \sum_{n=0}^{\infty} \frac{i^{2k} z^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{i^{2k+1} z^{2k+1}}{(2k+1)!}.$$

Since $i^{2k} = (i^2)^k = (-1)^k$ and $i^{2k+1} = i \cdot i^{2k} = i(-1)^k$, then

$$e^{iz} = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \cos z + i \sin z.$$

Then, we show that $e^{-iz} = \cos z - i \sin z$. We know that

$$\begin{aligned} e^{-iz} &= \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \sum_{n:\ \text{even}} \frac{(-iz)^n}{n!} + \sum_{n:\ \text{odd}} \frac{(-iz)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-iz)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(-iz)^{2k+1}}{(2k+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{2k} i^{2k} z^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(-1)^{2k+1} i^{2k+1} z^{2k+1}}{(2k+1)!}. \end{aligned}$$

Since $i^{2k} = (i^2)^k = (-1)^k$ and $i^{2k+1} = i \cdot i^{2k} = i(-1)^k$, then

$$e^{iz} = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} - i \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \cos z - i \sin z.$$

From the above two results, we can immediately get

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

II.

Solution

$$\frac{d}{dz} \cos z = \frac{d}{dz}\Big[\frac{1}{2}(e^{iz} + e^{-iz})\Big] = \frac{1}{2i}(e^{iz} - e^{-iz}) = -\sin z.$$

III.

Solution We have known that $\sin(z + \varsigma) = \sin z \cos \varsigma + \cos z \sin \varsigma$. On one hand,

$$\frac{d}{d\varsigma} \sin(z + \varsigma) = \cos(z + \varsigma);$$

one the other hand,

$$\frac{d}{d\varsigma}(\sin z \cos \varsigma + \cos z \sin \varsigma) = \sin z(-\sin \varsigma) + \cos z \cos \varsigma = \cos z \cos \varsigma - \sin z \sin \varsigma.$$

IV.

Solution Let

$$L = \limsup \sqrt[n]{|a_n|} = \inf_k \sup_{n \geq k} \sqrt[n]{|a_n|} = \lim_{k \to \infty} \sup\{\sqrt[k]{|a_k|}, \sqrt[k+1]{|a_{k+1}|}, \ldots\}.$$

Given $\varepsilon > 0$, there is an N such that $\sqrt[n]{|a_n|} < L + \varepsilon$ for all $n \geq N$. Suppose that

$$|z - z_0| < (1 - \varepsilon)/(L + \varepsilon),$$

then $|a_n||z - z_0|^n < (1 - \varepsilon)^n$ for all $n \geq N$, so $\sum_0^\infty a_n(z - z_0)^n$ is absolutely convergent. Hence, $R \geq (1 - \varepsilon)/(L + \varepsilon)$ for each $\varepsilon > 0$. Therefore, $R \geq 1/L$. Conversely, suppose that $|z - z_0| > 1/(L - \varepsilon)$ for each $\varepsilon > 0$. There are infinitely many indices n with $\sqrt[n]{|a_n|} > L - \varepsilon$. Hence, $|a_n||z - z_0|^n > 1$ for infinitely many indices n, so $\sum_0^\infty a_n(z - z_0)^n$ diverges. Hence, $R \leq 1/(L - \varepsilon)$ for each $\varepsilon > 0$, so $R \leq 1/L$. Hence, $R = 1/L$.