Key to Demonstration 7

I.
(a) 2
$$\log 2 = \log|2| + i\text{Arg}2 + i\cdot 2\pi k = \log|2| + i\cdot 2\pi k, \quad k\in\mathbb{Z};$$
(b) $-i$
$$\log(-i) = \log|-i| + i\text{Arg}(-i) + i\cdot 2\pi k = i(-\frac{\pi}{2} + 2\pi k), \quad k\in\mathbb{Z};$$
(c) $\sqrt{3}-i$
Since $\sqrt{3}-i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$,
$$\log(\sqrt{3}-i) = \log|\sqrt{3}-i| + i\text{Arg}(\sqrt{3}-i) + i\cdot 2\pi k = \log 2 + i\left(-\frac{\pi}{6} + 2\pi k\right), \quad k\in\mathbb{Z}.$$
II.
(a) 2
$$\text{Log}2 = \log|2| + i\text{Arg}2 = \log|2|;$$
(b) $-i$
$$\text{Log}(-i) = \log|-i| + i\text{Arg}(-i) = -\frac{\pi}{2}i;$$
(c) $\sqrt{3}-i$
$$\text{Log}(\sqrt{3}-i) = \log|\sqrt{3}-i| + i\text{Arg}(\sqrt{3}-i) = \log 2 - \frac{\pi}{6}i\,.$$
III.
(a) 1^π
Since $\log 1 = \log|1| + i\text{Arg}1 + i\cdot 2\pi k = i\cdot 2\pi k, \quad k\in\mathbb{Z}$,
$$1^\pi = e^{\pi\log 1} = e^{\pi\cdot i2\pi k} = e^{i2\pi^2 k}, \quad k\in\mathbb{Z};$$
(b) $1^{i\pi}$
$$1^{i\pi} = e^{i\pi\log 1} = e^{i\pi\cdot i2\pi k} = e^{-2\pi^2 k}, \quad k\in\mathbb{Z};$$
(c) i^i
Since $\log i = \log|i| + i\text{Arg}i + i\cdot 2\pi k = i\left(\frac{\pi}{2} + 2\pi k\right), \quad k\in\mathbb{Z}$,
$$i^i = e^{i\log i} = e^{i\cdot i(\frac{\pi}{2}+2\pi k)} = e^{-(\frac{\pi}{2}+2\pi k)}, \quad k\in\mathbb{Z}\,.$$
IV.
$$z^0 = e^{0\cdot\log z} = e^0 = 1.$$
V. (a)
$$\int\limits_{|z|=1} \frac{z}{(z-2)^2}dz = 0;$$

(b)

$$\int\limits_{|z|=2} \frac{e^z}{z(z-3)}dz = -\frac{1}{3}\int\limits_{|z|=2} \frac{e^z}{z}dz + \frac{1}{3}\int\limits_{|z|=2} \frac{e^z}{z-3}dz = -\frac{1}{3}\cdot 2\pi i \cdot e^0 + 0 = -\frac{2\pi i}{3};$$

(c)

$$\begin{aligned}\int\limits_{|z+1|=2} \frac{z^2}{4-z^2}dz &= \int\limits_{|z+1|=2} -1 dz - 4\int\limits_{|z+1|=2} \frac{1}{z^2-4}dz \\ &= -4\pi + \int\limits_{|z+1|=2} \frac{1}{z+2}dz - \int\limits_{|z+1|=2} \frac{1}{z-2}dz \\ &= -4\pi + 2\pi i + 0 = -4\pi + 2\pi i;\end{aligned}$$

(d)

Since we know that $\frac{\sin z}{z} \to 1$ as $z \to 0$, and $\frac{\sin z}{z}$ is analytic in $\mathbb{D}\setminus 0$, where $\mathbb{D} = \{z : |z| < 1\}$,

$$\int\limits_{|z|=1} \frac{\sin z}{z}dz = 0\,.$$

VI. Let $z = \cos t + i\sin t$, $dt = \frac{dz}{iz}$ and $\cos t = \frac{1}{2}\Big(z + \frac{1}{z}\Big)$, then,

$$\begin{aligned}\int_0^\pi \frac{d\theta}{1+\sin^2\theta} &= \int_0^\pi \frac{d\theta}{1+\frac{1-\cos 2\theta}{2}} = \int_0^\pi \frac{2d\theta}{3-\cos 2\theta} \\ &= \int_0^{2\pi} \frac{dt}{3-\cos t} = 2i\int\limits_{|z|=1} \frac{dz}{z^2-6z+1} = \frac{2i}{4\sqrt{2}}\int\limits_{|z|=1}\left(\frac{1}{z-(3+2\sqrt{2})} - \frac{1}{z-(3-2\sqrt{2})}\right)dz \\ &= \frac{2i}{4\sqrt{2}}\int\limits_{|z|=1} \frac{1}{z-(3+2\sqrt{2})}dz - \frac{2i}{4\sqrt{2}}\int\limits_{|z|=1} \frac{1}{z-(3-2\sqrt{2})}dz = \frac{2i}{4\sqrt{2}}\cdot 0 - \frac{2i}{4\sqrt{2}}\cdot 2\pi i = \frac{\pi}{\sqrt{2}}\,.\end{aligned}$$