Key to Demonstration 8

I.

Solution. (a) Since

$$\frac{e^z}{z-\alpha} = \frac{e^\alpha e^{z-\alpha}}{z-\alpha} = \frac{e^\alpha}{z-\alpha} + e^\alpha \sum_1^\infty \frac{(z-\alpha)^{n-1}}{n!} = \frac{e^\alpha}{z-\alpha} + g(z),$$

where $g(z)$ is entire (because $\rho = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty}(n+1) = +\infty$),

$$\int_C \frac{e^z}{z-\alpha}dz = e^\alpha \int_C \frac{1}{z-\alpha}dz + \int_C g(z)dz = e^\alpha \cdot 2\pi i + 0 = 2\pi i e^\alpha.$$

(b) Since

$$\frac{e^z}{(z-\alpha)^n} = \frac{e^\alpha e^{z-\alpha}}{(z-\alpha)^n} = \frac{e^\alpha}{(z-\alpha)^n} \sum_{k=0}^\infty \frac{(z-\alpha)^k}{k!} = e^\alpha \sum_{k=0}^\infty \frac{1}{k!(z-\alpha)^{n-k}}$$
$$= e^\alpha \Big(\frac{1}{(z-\alpha)^n} + \frac{1}{(z-\alpha)^{(n-1)}} + \frac{1}{(z-\alpha)^{(n-2)}} + \cdots + \frac{1}{(z-\alpha)^1} + g(z) \Big),$$

for some $n \geq 2$, and $g(z)$ is entire, we have

$$\int_C \frac{e^z}{(z-\alpha)^n}dz$$
$$= e^\alpha \Big(\int_C \frac{1}{(z-\alpha)^n}dz + \int_C \frac{1}{(z-\alpha)^{(n-1)}}dz + \int_C \frac{1}{(z-\alpha)^{(n-2)}}dz + \cdots + \int_C \frac{1}{z-\alpha}dz + \int_C g(z)dz \Big)$$
$$= e^\alpha (0 + 0 + 0 + \cdots + 2\pi i + 0) = 2\pi i e^\alpha.$$

II.

Solution. (a) $\gamma : |z| = 2$. Decomposing the integral by partial fractions, we obtain

$$\int_\gamma \frac{\cos z}{z^3+z}dz = \int_\gamma \frac{\cos z}{z}dz - \frac{1}{2}\int_\gamma \frac{\cos z}{z+i}dz - \frac{1}{2}\int_\gamma \frac{\cos z}{z-i}dz$$
$$= 2\pi i \Big[\cos(0) - \frac{1}{2}\cos(-i) - \frac{1}{2}\cos i \Big] = 2\pi i[1 - \cosh(1)].$$

(b) $\gamma : |z| = \frac{1}{2}$. As $\cos z/(z^2+1)$ is analytic on and inside γ, the integral equals $2\pi i$ times its value at $z = 0$, that is,

$$\int_\gamma \frac{\cos z}{z^3+z}dz = 2\pi i.$$

(c) $\gamma : |z - i/2| = 1$. Since $\cos z/(z+i)$ is analytic on and inside γ, by partial fractions we have

$$\frac{1}{z(z-i)} = i\Big(\frac{1}{z} - \frac{1}{z-i}\Big),$$

so that

$$\int_\gamma \frac{\cos z}{z^3+z} dz = 2\pi i\Big[i\Big(\frac{\cos(0)}{i}\Big) - i\Big(\frac{\cos i}{2i}\Big)\Big] = 2\pi i\Big[1 - \frac{1}{2}\cosh(1)\Big].$$

Of course, we can do all three examples utilizing the partial fraction decomposition in part (a), since the corresponding integrals vanish when the points 0 or $\pm i$ lie outside γ.

III.

Proof. Let $C : |\xi| = R$, for any $z \in I(C)$, according to Theorem 3.2 d)

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int\limits_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi,$$

then we have

$$|f^{(n)}(z)| = \Big|\frac{n!}{2\pi i}\int\limits_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\Big| \le \frac{n!}{2\pi}\int\limits_C \frac{|f(\xi)|}{(|\xi| - |z|)^{n+1}} |d\xi| \le \frac{n!}{2\pi}\int\limits_C \frac{|f(\xi)|}{(R - |z|)^{n+1}} |d\xi|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{(R-|z|)^{n+1}}\int\limits_C |d\xi| = \frac{n!}{2\pi}\cdot\frac{M}{(R-|z|)^{n+1}}\cdot 2\pi R = \frac{MRn!}{(R-|z|)^{n+1}}.$$

IV.

Solution. Denote γ : the semicircle from -1 to 1 passing through i, and denote ι : be the straight segment from 1 to -1, by Cauchy Theorem, we know that

$$\int\limits_{\gamma+\iota} e^z dz = 0,$$

thus

$$\int\limits_\gamma e^z dz = -\int\limits_\iota e^z dz = \int\limits_{\iota^-} e^z dz = \int\limits_\iota e^x dx = \int_{-1}^1 e^x dx = e - e^{-1}.$$

V.

Prove. By Theorem 3.2 a), we know that when z is inside of the circle C with radius r,

$$f(z) = \frac{1}{2\pi i}\int\limits_C \frac{f(\xi)}{\xi - z} d\xi,$$

if z is the center of the circle, $\xi = z + re^{it}$, then

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{it})}{re^{it}} rie^{it} dt = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it}) dt.$$

VI.

Solution.

$$
\begin{aligned}
\frac{\sin z}{z-\alpha} &= \frac{\frac{1}{2i}(e^{iz}-e^{-iz})}{z-\alpha} = \frac{1}{2i}\frac{1}{z-\alpha}\big(e^{i\alpha}e^{i(z-\alpha)} - e^{-i\alpha}e^{-i(z-\alpha)}\big) \\
&= \frac{1}{2i}\frac{1}{z-\alpha}\Big(e^{i\alpha}\sum_{n=0}^{\infty}\frac{(i(z-\alpha))^{n-1}}{n!} - e^{-i\alpha}\sum_{n=0}^{\infty}\frac{(-i(z-\alpha))^{n-1}}{n!}\Big) \\
&= \frac{e^{i\alpha}}{2i}\sum_{n=0}^{\infty}\frac{i^n(z-\alpha)^{n-1}}{n!} - \frac{e^{-i\alpha}}{2i}\sum_{n=0}^{\infty}\frac{(-i)^n(z-\alpha)^{n-1}}{n!} \\
&= \frac{e^{i\alpha}}{2i}\Big(\frac{1}{z-\alpha} + g_1(z)\Big) - \frac{e^{-i\alpha}}{2i}\Big(\frac{1}{z-\alpha} + g_2(z)\Big),
\end{aligned}
$$

thus we can get

$$
\begin{aligned}
\int\limits_C \frac{\sin z}{z-\alpha}dz &= \frac{e^{i\alpha}}{2i}\Big(\int\limits_C \frac{dz}{z-\alpha} + \int\limits_C g_1(z)dz\Big) - \frac{e^{-i\alpha}}{2i}\Big(\int\limits_C \frac{dz}{z-\alpha} + \int\limits_C g_2(z)dz\Big) \\
&= \frac{e^{i\alpha}}{2i}\cdot 2\pi i - \frac{e^{-i\alpha}}{2i}\cdot 2\pi i = \pi(e^{i\alpha} - e^{-i\alpha}).
\end{aligned}
$$