Key to Demonstration 9

I.

Proof.

$$e^{f(z)} = e^{u(z)+iv(z)} = e^{u(z)}e^{iv(z)} \Rightarrow \left|e^{f(z)}\right| = \left|e^{u(z)}\right|\left|e^{iv(z)}\right| = e^{u(z)} \leq e^{M} < \infty,$$

since $e^{f(z)}$ is entire, by Liouville Theorem, $e^{f(z)} \equiv C(\neq 0)$, then by Theorem 2.7 f),

$$f(z) = \log C + 2k\pi i, k \in \mathbb{Z},$$

thus $f(z)$ is constant.

II.

Solution. (a)

n	at z	at 1	coefficients
0	z^{-1}	1	1
1	$-z^{-2}$	-1	-1
2	$2z^{-3}$	2	$\frac{2}{2!} = 1$
3	$-2 \cdot 3z^{-4}$	$-2 \cdot 3$	$\frac{-2\cdot 3}{3!} = -1$
4	$2 \cdot 3 \cdot 4z^{-5}$	$2 \cdot 3 \cdot 4$	$\frac{2\cdot 3\cdot 4}{4!} = 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$(2k!)z^{-2k-1}$	$(2k)!$	1
$2k+1$	$-(2k+1)!z^{-2k-2}$	$-(2k+1)!$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The series we want is then

$$f(z) = 1 - (z-1) + (z-1)^2 - \cdots + (z-1)^{2k} - (z-1)^{2k+1} + \cdots$$

and by geometric series, when $|1-z| < 1$, we can also get

$$f(z) = \frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$

(b)

n	at z	at 2	coefficients
0	z^{-1}	2^{-1}	2^{-1}
1	$-z^{-2}$	-2^{-2}	-2^{-2}
2	$2z^{-3}$	$2 \cdot 2^{-3}$	$\frac{2\cdot 2^{-3}}{2!} = 2^{-3}$
3	$-2 \cdot 3z^{-4}$	$-2 \cdot 3 \cdot 2^{-4}$	$\frac{-2\cdot 3\cdot 2^{-4}}{3!} = -2^{-4}$
4	$2 \cdot 3 \cdot 4z^{-5}$	$2 \cdot 3 \cdot 4 \cdot 2^{-5}$	$\frac{2\cdot 3\cdot 4\cdot 2^{-5}}{4!} = 2^{-5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$(2k!)z^{-2k-1}$	$(2k)!2^{-2k-1}$	2^{-2k-1}
$2k+1$	$-(2k+1)!z^{-2k-2}$	$-(2k+1)!2^{-2k-2}$	-2^{-2k-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The series we want is then

$$f(z) = 2^{-1} - 2^{-2}(z-1) + 2^{-3}(z-1)^2 - 2^{-4}(z-2)^3 + 2^{-5}(z-2)^4 \cdots + 2^{-2k-1}(z-1)^{2k} - 2^{-2k-2}(z-1)^{2k+1} + \cdots$$

and by geometric series, when $|z-2| < 1$, we can also get

$$f(z) = \frac{1}{z} = \frac{1}{2-(2-z)} = \frac{1}{2}\sum_{n=0}^{\infty}\Big(-\frac{1}{2}(z-2)\Big)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n.$$

(c) Denote $w = 1+z$, we get

$$\begin{aligned}
\mathrm{Log} w &= \int_1^w \frac{d\varsigma}{\varsigma} = \int_1^w \big[1-(\varsigma-1)+(\varsigma-1)^2-(\varsigma-1)^3+\cdots+(\varsigma-1)^{2k}-(\varsigma-1)^{2k+1}+\cdots\big]d\varsigma \\
&= \int_1^w d\varsigma - \int_1^w (\varsigma-1)d\varsigma + \int_1^w (\varsigma-1)^2 d\varsigma - \int_1^w (\varsigma-1)^3 d\varsigma + \cdots + \int_1^w (\varsigma-1)^{2k} d\varsigma - \int_1^w (\varsigma-1)^{2k+1} d\varsigma + \cdots \\
&= w-1-\frac{(w-1)^2}{2}+\frac{(w-1)^3}{3}-\frac{(w-1)^4}{4}+\cdots+\frac{(w-1)^{2k+1}}{2k+1}-\frac{(w-1)^{2k+2}}{2k+2}+\cdots \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(w-1)^k}{k} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}.
\end{aligned}$$

III.

Solution. We compute derivatives and evaluate at $\pi/2$:

n	at z	at $\pi/2$	coefficients
0	$\sin z$	1	1
1	$\cos z$	0	0
2	$-\sin z$	-1	$-1/2!$
3	$-\cos z$	0	0
4	$\sin z$	1	$1/4!$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$(-1)^k \sin z$	$(-1)^k$	$(-1)^k/(2k)!$
$2k+1$	$(-1)^k \cos z$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The series we want is then

$$\sin z = 1 - \frac{1}{2!}\Big(z-\frac{\pi}{2}\Big)^4 + \cdots + \frac{(-1)^k}{(2k)!}\Big(z-\frac{\pi}{2}\Big)^{2k} + \cdots = \sum_{0}^{\infty}\frac{(-1)^k}{(2k)!}\Big(z-\frac{\pi}{2}\Big)^{2k}.$$

IV.

Solution. (a) Since $|z| > 1$, $\left|\frac{1}{z}\right| < 1$, by geometric series, we have

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{k=0}^{\infty}\Big(\frac{1}{z}\Big)^k = -\sum_{k=0}^{\infty}\Big(\frac{1}{z}\Big)^{k+1}.$$

(b) We know that for $z \in \mathbb{C}$,

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

so

$$\frac{\sin z}{z^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-2}.$$

V.

Proof. For $r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})} = \int_0^{2\pi} \Big(\sum_0^{\infty} a_n r^n e^{in\theta} \Big) \Big(\sum_0^{\infty} \bar{a}_k r^k e^{-ik\theta} \Big) d\theta$$

$$= 2\pi \sum_0^{\infty} |a_n|^2 r^{2n},$$

on the other side, we know that $|f(z)| \leq M$, so

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq M^2 \cdot 2\pi.$$

Then for any $r < 1$, we have

$$2\pi \sum_0^{\infty} |a_n|^2 r^{2n} \leq M^2 \cdot 2\pi,$$

let $r \to 1$, we get

$$\sum_0^{\infty} |a_n|^2 \leq M^2.$$