Key to Demonstration 10

I.

Solution. (a) We compute derivatives and evaluate at $\pi/2$:

n	at z	at $\pi/2$	coefficients
0	$\cos z$	0	0
1	$-\sin z$	-1	-1
2	$-\cos z$	0	0
3	$\sin z$	1	$1/3!$
4	$\cos z$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k-1$	$(-1)^k \sin z$	$(-1)^k$	$(-1)^k/(2k-1)!$
$2k$	$(-1)^k \cos z$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The series we want is then

$$\cos z = -\Big(z-\frac{\pi}{2}\Big)+\frac{1}{3!}\Big(z-\frac{\pi}{2}\Big)^3-\cdots+\frac{(-1)^k}{(2k-1)!}\Big(z-\frac{\pi}{2}\Big)^{2k-1}+\cdots = \sum_0^\infty \frac{(-1)^k}{(2k-1)!}\Big(z-\frac{\pi}{2}\Big)^{2k-1}.$$

(b) Since $\cos^2 z = \frac{1}{2}\cos 2z + \frac{1}{2}$, by geometric series, we get

$$\cos^2 z = \frac{1}{2} + \frac{1}{2}\sum_0^\infty \frac{(-1)^k}{(2k)!}(2z)^{2k} = 1 + \sum_1^\infty \frac{(-4)^k}{(2k)!} z^{2k}.$$

II.

Solution. (a) Since $0 < |z-1| < 1$, by geometric series, we get

$$\frac{1}{z(1-z)} = \frac{1}{1-z}\cdot\frac{1}{1+(z-1)} = \frac{1}{1-z}\cdot\sum_{k=0}^\infty[-(z-1)]^k = \sum_{k=0}^\infty (-1)^{k+1}(z-1)^{k-1}.$$

(b) Since $|z-1| > 1$, $\frac{1}{|z-1|} < 1$, by geometric series, we get

$$\begin{aligned}\frac{1}{z(1-z)} &= \frac{1}{1-z}\cdot\frac{1}{1+(z-1)} = \frac{1}{1-z}\cdot\frac{1}{(z-1)(1+\frac{1}{z-1})}\\ &= -\frac{1}{(z-1)^2}\cdot\sum_{k=0}^\infty\Big(-\frac{1}{z-1}\Big)^k = \sum_{k=0}^\infty(-1)^{k+1}\Big(\frac{1}{z-1}\Big)^{k+2}.\end{aligned}$$

III.

Solution. By geometric series, when $|z| > 0$, we have

$$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\Big(1-\sum_{k=0}^\infty\frac{(-1)^k}{(2k)!}z^{2k}\Big) = \sum_{k=1}^\infty\frac{(-1)^{k+1}}{(2k)!}z^{2k-2}.$$

From the series above, it is easy to see that $\dfrac{1-\cos z}{z^2} \to \dfrac{1}{2}$, as $z \to 0$.

IV.

Proof. Let $z = \cos\theta + i\sin\theta$, $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}(z + \dfrac{1}{z})$, then we can get

$$\int_0^{2\pi} e^{2\cos\theta} d\theta = -\,i \int_{|z|=1} \frac{e^{z+\frac{1}{z}}}{z} dz = -i \int_{|z|=1} \frac{e^z}{z} \sum_{n=0}^{\infty} \frac{1}{n! z^n} dz = -i \sum_{n=0}^{\infty} \frac{1}{n!} \int_{|z|=1} \frac{e^z}{z^{n+1}} dz$$
$$= -\,i \sum_{n=0}^{\infty} \frac{1}{n!} \frac{2\pi i}{n!} \cdot \left(e^z\right)^{(n)}\Big|_{z=0} = 2\pi \sum_{n=0}^{\infty} \frac{1}{(n!)^2}.$$

V.

Proof. (a) It is well known to all that for any $\alpha \in \mathbb{R}$, $\cos^2\alpha + \sin^2\alpha = 1$. Now let $f(z) = \cos^2 z$, and let $g(z) = 1 - \sin^2 z$, it is easy to see that $f(z)$ and $g(z)$ are all holomorphic in $\mathbb{C}$. Let $\{\alpha_n\}$ be any convergent real number sequence in $\mathbb{C}$, thus we know that $f(\alpha_n) = g(\alpha_n)$, $n = 1, 2, \cdots$, by uniqueness (identity) theorem, we have $f(z) = g(z)$, that is $\cos^2 z + \sin^2 z = 1$.

(b) For $\alpha \in \mathbb{R}$, let $f(z) = \sin(z+\alpha)$ and $g(z) = \sin z \cos\alpha + \cos z \sin\alpha$. The proof is similar with the above.