Key to Demonstration 11

I.

Proof. (**Note**: The manner of one-to-one indicates the mapping is injective and surjective as well.)

For two points w_1, w_2 in w-plane, be not ∞, if $w_1 = w_2$, it means $\alpha z_1 + \beta = \alpha z_2 + \beta$, then $z_1 = z_2$, since $\alpha \neq 0$, so it is injective.

For the point $w_0 = re^{i\theta_0}$ in w-plane, be not ∞, $z_0 = \dfrac{re^{i\theta_0}}{\alpha} - \dfrac{\beta}{\alpha}$, where $\alpha \neq 0$, is in z-plane, such that $z_0 \to w_0$ under this mapping, so it is surjective.

For $z = \infty$, choose any sequence $\{z_n\} \to \infty$, it means that for any $r > 0$, there exists $N \in \mathbb{N}$, such that when $n \geq N$, $\{z_n\} \subset B(\infty, r)$, where

$$B(\infty, r) = \{z \in \mathbb{Z}, |z| > r\}.$$

Under the linear transformation $w = \alpha z + \beta$, $\alpha \neq 0$, the chosen sequence is first mapped to the sequence $\{\eta_n\} = \{\alpha z_n\}$. It is obvious that when $n \geq N$, $\{\eta_n\} \subset B(\infty, r|\alpha|)$, that is $\{\eta_n\} \to \infty$ too. Then for $\{w_n\} = \{\alpha z + \beta\} \subset B\big(\infty, \big|r|\alpha| - |\beta|\big|\big)$, when $n \geq N$, it tells us $\{w_n\} \to \infty$. It showes that the mapping maps $z = \infty$ onto $w = \infty$. Thus finishing the proof.

II.

Proof. If $\alpha z + \beta = 0$, then $w(z)$ reduces to constant 0; if $\alpha z + \beta \neq 0$, let $\delta = \dfrac{\beta\gamma}{\alpha}$, then

$$w(z) = \frac{\alpha z + \beta}{\gamma z + \frac{\beta\gamma}{\alpha}} = \frac{\alpha(\alpha z + \beta)}{\alpha\gamma z + \beta\gamma} = \frac{\alpha}{\gamma}\,,$$

is also a constant.

III.

Proof. (**Note**: The manner of one-to-one indicates the mapping is injective and surjective as well.)

For two points w_1, w_2 in w-plane, be not 0 nor ∞, if $w_1 = w_2$, it means $\dfrac{1}{z_1} = \dfrac{1}{z_2}$, then $z_1 = z_2$, so it is injective.

For the point $w_0 = re^{i\theta_0}$ in w-plane, be not 0 nor ∞, $z_0 = r^{-1}e^{-i\theta_0}$ is in z-plane, such that $z_0 \to w_0$ under this mapping, so it is surjective.

With the conventions, if $w_0 = 0$, $z_0 = 1/0 = \infty$; if $w_0 = \infty$, $z = 1/\infty = 0$, so this $w = 1/z$ maps the extended z-plane in a one-to-one manner onto the extended w-plane.

IV.

Proof. A Möbius transformation $w(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, where $\alpha\delta - \beta\gamma \neq 0$, is a linear fractional mapping. If $\gamma = 0$, the Möbius transformation reduces to

$$w = \frac{\alpha}{\delta} z + \frac{\beta}{\delta},$$

which is linear, so is one-to-one mapping from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$, analytic. If $\gamma \neq 0$,

$$w(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = \frac{\beta\gamma - \alpha\delta}{\gamma} \cdot \frac{1}{\gamma z + \delta} + \frac{\alpha}{\gamma}, \tag{1}$$

it is easy to see that this transformation is analytic except when $\gamma z+\delta=0$, i.e. $z=-\dfrac{\delta}{\gamma}$. To see this transformation is one-to-one, we set

$$\begin{cases} z_1 = \gamma z+\delta \\ z_2 = 1/z_1 \\ w = \big[(\beta\gamma-\alpha\delta)/\gamma\big] z_2 + \big(\alpha/\gamma\big) \end{cases}$$

we see that (1) is a linear transformation followed by an inversion followed in turn by another linear transformation.

V.

Proof. Choose a circle from (X), denoted by C_1, and a circle from (Y), denoted by C_2. Let's see when and where the two circles intersect.

$$\begin{cases} \left(u-\dfrac{1}{2a}\right)^2 + v^2 = \left(\dfrac{1}{2a}\right)^2 \\ u^2 + \left(v+\dfrac{1}{2b}\right)^2 = \left(\dfrac{1}{2b}\right)^2. \end{cases}$$

Immediately, we can get

$$\begin{cases} u^2 - \dfrac{u}{a} + v^2 = 0 \\ u^2 + v^2 + \dfrac{v}{b} = 0. \end{cases}$$

By simple calculation, we get they intersect at two points $(0,0)$ and $\left(\dfrac{a}{a^2+b^2}, -\dfrac{b}{a^2+b^2}\right)$.

It is easy to see that C_1 and C_2 intersects at $(0,0)$ in right angle. We only need to consider the other intersect point.

For C_1, let $F(u,v)=u^2-\dfrac{u}{a}+v^2$, the slope of the tangent line of C_1 is

$$\frac{du}{dv} = \frac{\frac{\partial F}{\partial v}}{\frac{\partial F}{\partial u}} = \frac{2v}{2u-\frac{1}{a}},$$

then the slope of tangent for C_1 at the point $\left(\dfrac{a}{a^2+b^2}, -\dfrac{b}{a^2+b^2}\right)$ is $\dfrac{-2ab}{a^2-b^2}$. In the same way, we get the slope of tangent for C_2 at the point $\left(\dfrac{a}{a^2+b^2}, -\dfrac{b}{a^2+b^2}\right)$ is $\dfrac{a^2-b^2}{2ab}$. Obviously, the two tangent intersects in right angle, since the product of the two slope

$$\frac{-2ab}{a^2-b^2} \cdot \frac{a^2-b^2}{2ab} = -1.$$

Thus finishing the proof.

VI.

Proof.

$$T\circ S(z) = T(S(z)) = \frac{\alpha_1(S(z))+\beta_1}{\gamma_1(s(z))+\delta_1} = \frac{\alpha_1\left(\frac{\alpha_2 z+\beta_2}{\gamma_2 z+\delta_2}\right)+\beta_1}{\gamma_1\left(\frac{\alpha_2 z+\beta_2}{\gamma_2 z+\delta_2}\right)+\delta_1} = \frac{(\alpha_1\alpha_2+\beta_1\gamma_2)z+\alpha_1\beta_2+\beta_1\delta_2}{(\alpha_2\gamma_1+\gamma_2\delta_1)z+\beta_2\gamma_1+\delta_1\delta_2}.$$

Let $\alpha = \alpha_1\alpha_2 + \beta_1\gamma_2$, $\beta = \alpha_1\beta_2 + \beta_1\delta_2$, $\gamma = \alpha_2\gamma_1 + \gamma_2\delta_1$ and $\delta = \beta_2\gamma_1 + \delta_1\delta_2$, then we finally get

$$T \circ S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$