Ratkaisu

Etsitään siis pisteiden $(x_0, f(x_0)) = (-2, 2)$, $(x_1, f(x_1)) = (-1, 1)$, $(x_2, f(x_2)) = (1, 1)$ ja $(x_3, f(x_3)) = (3, 2)$ kautta kulkevaa 3. asteen polynomia $P_3(x)$ muodossa

$$P_3(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + (x - x_0)(x - x_1)(x - x_2)a_3,$$

jolle pätee

$$P_3(x_0) = a_0 = f(x_0) := f[x_0],$$

$$P_3(x_1) = a_0 + a_1(x_1 - x_0) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$\Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} := f[x_0, x_1].$$

Edelleen

$$\begin{aligned} P_3(x_2) &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &= f(x_0) + \tfrac{f(x_1)-f(x_0)}{x_1-x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) \end{aligned}$$

$$\Rightarrow a_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} := f[x_0, x_1, x_2].$$

Vastaavasti

$$\begin{aligned} P_3(x_3) &= a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) \\ &+ a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = f(x_3) \end{aligned}$$

$$\Rightarrow a_3 = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} := f[x_0, x_1, x_2, x_3].$$

Yleisesti pätee

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

ja Newtonin *jaettujen erotusten* avulla muodostettu inteprolaatiopolynomi on muotoa

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1) \ldots (x - x_{k-1}).$$

Polynomin kertoimet $a_k = f[x_0, x_1, \ldots, x_k]$ on tapana laskea jaettujen erotusten kaavio avulla

$$\begin{array}{llll}
x_0 & f[x_0] & & \\
x_1 & f[x_1] & f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} & \\
x_2 & f[x_2] & f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} & f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} \\
x_3 & f[x_3] & f[x_2,x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} & f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} \quad f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}
\end{array}$$

Esimerkin tapauksessa kaavio saadaan muotoon

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	-2.0	2.0			
1	-1.0	1.0	-1.0		
2	1.0	1.0	0.0	1/3	
3	3.0	2.0	1/2	1/8	-1/24

Poimimalla polynomin $P_3(x)$ kertoimet saadaan

$$\begin{aligned}
P_3(x) &= 2.0 - 1.0(x+2.0) + 1/3(x+2.0)(x+1.0) - 1/24(x+2.0)(x+1.0)(x-1.0) \\
&= \frac{3}{4} + \frac{1}{24}x + \frac{1}{4}x^2 - \frac{1}{24}x^3.
\end{aligned}$$