Numerical linear algebra autumn 2004

Problem set 10

1. Consider the problem $Ax=b$. The following method is known as *Richardson iteration:*

$$x^{m+1}=x^m+\alpha r^m$$

where r^m is the residual and α is some parameter.

   - write the residual as $r^m=p(A)b$. Which polynomial p do you get?
   - if you have some information about eigenvalues of A how would you choose the parameter α ?
   - Explain how this is a slight generalisation of fixed point iteration.

2. Let us consider problems $Ax = b$ and $UAU^*y = Ub$ where U is unitary. Let us apply GMRES to both problems. We obtain two sequences of residuals: r^m and $\tilde{r}^m$. How are r^m and $\tilde{r}^m$ related?

3. Compare the performance of the GMRES and restarted GMRES in Matlab. Choose big enough matrices (sparse and dense). Compare the convergence rates and the required computer time. In which cases the restarted version does not converge at all? To have different test cases you can first choose some random V and Λ and then put $A=V\Lambda V^{-1}$. In this case A will be dense.

4. Let A be symmetric and positive definite and define an inner product:

$$\langle x\,,y\,\rangle_A=\langle x\,,Ay\,\rangle$$

Vectors x and y are said to be A–orthogonal, if $\langle x\,,y\,\rangle_A=0$. Show that if a set of vectors are A–orthogonal to each other, then they are linearly independent.

5. Conjugate gradient method for solving $Ax=b$ is based on minimising the function

$$\varphi(x)=\frac{1}{2}\langle x\,,Ax\,\rangle-\langle b\,,x\,\rangle$$

Why doesn't this idea work for nonsymmetric problems?

Homework

Matlab's command qmr stands for *quasi minimal residual*. It is a modification of the GMRES method which avoids the problem of storing the whole Q_m matrix which contains the orthonormal basis of the Krylov subspace. This makes the computation of GMRES expensive for large values of m, and hence makes restarting necessary in some cases. In QMR there is no need to restart, but as the name implies the minimum property of the GMRES is not guaranteed, and hence the convergence is not monotone in general. Compare the convergence of GMRES and QMR methods. Compare also the time required.