Numerical linear algebra autumn 2004

Problem set 11

1. Let A be real, symmetric and positive definite and let us define

$$\varphi(x) = \frac{1}{2}\langle x, Ax \rangle - \langle b, x \rangle$$

One natural idea to minimise this is to take gradient as search direction: hence at step m we set $p^m = -\nabla\varphi(x^m) = -(Ax^m - b) = r^m$. This is called the method of *steepest descent.* Examine the convergence in the case

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 20 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

How many iterations are needed to get a reasonable accuracy? Why is the convergence slow? Recall that conjugate gradient gives the solution in 2 iterations. Check this.

2. Let A be real and symmetric 805×805 matrix with eigenvalues

$$1, 1.01, 1.02, \ldots, 8.99, 9, 10, 12, 16, 24$$

How many iterations of conjugate gradient is needed so that the initial error $\|e^0\|_A$ is reduced by a factor of 10^6?

3. Conjugate gradient iteration is applied to a symmetric positive definite matrix. We know that $\|e^0\|_A = 1$ and $\|e^{10}\|_A = 2^{-9}$. Based on this data:

   - what bound can you give on $\kappa_2(A)$?
   - what bound can you give on $\|e^{20}\|_A$?

4. One can use conjugate gradient method for general problem $Ax = b$ by solving the normal equations $A^T Ax = A^T b$.

   - Let us consider

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

   Show that GMRES applied to original problem takes n steps while conjugate gradient & normal equations converge in one step.

   - Construct an example where conjugate gradient & normal equations need n steps, but GMRES converges in 2 steps.

   Hint: consider first a certain defective 4×4 matrix and then generalise this to matrices of arbitrary size.

5. Let A be a $n \times n$ tridiagonal matrix such that $a_{ii} = i$ and $a_{i,i+1} = a_{i,i-1} = 1$. Take various n and compute the solution with conjugate gradient with Matlab's command `pcg`. Plot the residuals. Is the convergence monotone? Compare the residual curve to the theoretical convergence curve

$$2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^m$$

Note that this is really estimate for the error in A–norm and not for the residual. Does it anyway seem to give a reasonable estimate for the residual?