Numerical linear algebra autumn 2004

Problem set 2

1. Consider the following matrix

$$A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}$$

   Compute by hand the (real) singular value decompostion $A = U\Sigma V^T$. What is $\|A\|_p$ for $p = 1, 2, \infty, F$? Compute A^{-1} using SVD.

2. Compute the eigenvalues of A (A as in the previous problem) . Check that

$$\det(A) = \lambda_1 \lambda_2 \quad \text{and} \quad |\det(A)| = \sigma_1 \sigma_2$$

   What is the area of ellipsoid onto which A maps the unit disk of $\mathbb{R}^2$? What is the best rank one approximation of A ?

3. By eigenvalue decomposition of a square matrix A is meant a factorization of the form $A = X\Lambda X^{-1}$ where the columns of X are the eigenvectors and Λ is a diagonal matrix with eigenvalues on the diagonal. Give an example of a 2×2 matrix which does not have an eigenvalue decomposition. Suppose that $A \in \mathbb{C}^{m \times m}$ and $A = U\Sigma V^*$. What is an eigenvalue decomposition of

$$B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} ?$$

4. Suppose that $A \in \mathbb{C}^{m \times n}$ and $A = U\Sigma V^*$. What are the eigenvalue decompositions of AA^* and A^*A ?

5. A and B are unitarily equivalent if there is a unitary matrix Q such that $A = QBQ^*$. Is the following statement true or false:

   - A and B are unitarily equivalent if and only if A and B have same singular values.

   Prove this or give counterexamples.

Homework (to be returned to Katya before the exam)

Splines.

Let $0 = x_0 < x_1 < \ldots < x_{n+1} = 1$. For simplicity let us assume that $x_i = ih$ where $h = 1/(n+1)$. Let us be given some points (x_i, y_i) , $i = 0, \ldots, n+1$, and define the intervals $I_k = [x_{k-1}, x_k]$. We want to construct a function (cubic spline) $s: [0,1] \to \mathbb{R}$ with the following properties:

I. s is twice continuously differentiable

II. $s(x_i) = y_i$ for all $i = 0, \ldots, n+1$

III. s restricted to each I_k is a polynomial of third degree

IV. $s'(0) = s'(1) = 0$

Since a polynomial of third degree is specified by 4 coefficients and there are $n+1$ intervals there are in principle $4n+4$ unknowns to be determined. However, it is in fact enough to solve a linear system with n unknowns. Let us consider following polynomials

$$\begin{aligned}
\varphi_1(x) &= 1-3x^2+2x^3 \\
\varphi_2(x) &= 3x^2-2x^3 \\
\varphi_3(x) &= x-2x^2+x^3 \\
\varphi_4(x) &= -x^2+x^3
\end{aligned}$$

See the picture.

- Any 3rd degree polynomial in interval I_k can be represented as

$$p_k(x) = a_k\,\varphi_1(\frac{x}{h}-k+1) + b_k\,\varphi_2(\frac{x}{h}-k+1) + h\,c_k\,\varphi_3(\frac{x}{h}-k+1) + h\,d_k\,\varphi_4(\frac{x}{h}-k+1)$$

  This is our condition III.

- Now show that the condition II implies that $a_k = y_{k-1}$ and $b_k = y_k$. With this choice our s is already continuous.

- Let us denote $z_k = s'(x_k)$. Obviously requirement IV means that $z_0 = z_{n+1} = 0$. Now show that requiring the first derivative of s to be continuous (part of the condition I) implies that $c_k = z_{k-1}$ and $d_k = z_k$.

- Let us further put $z = (z_1, \ldots, z_n)$. We now have to compute z. Requiring that the second derivative of s is continuous gives us equations which determine z. Formulate the problem in the form $Az = v$. What kind of properties does A have ? Is it symmetric? Is it sparse (harva) ? Does the system always has a solution ? Think about various criteria which guarantee the existence of a unique solution. Can some of them be applied in this case?