Numerical linear algebra autumn 2004

Problem set 3 Exercises in class M352

1. Let $P \in \mathbb{C}^{m \times m}$ be a (nonzero) projection matrix. Show that $\|P\|_2 \geq 1$. Then show that

$$\|P\|_2 = 1 \quad \Longleftrightarrow \quad P \text{ is an orthogonal projector}$$

2. Let $A, B \in \mathbb{R}^{3 \times 2}$ with $\text{rank}(A) = \text{rank}(B) = 2$. We want to find a vector $v \neq 0$ such that

$$v \in \text{im}(A) \cap \text{im}(B)$$

   How can this be done by doing 3 QR decompositions? Explain the algorithm also geometrically.

3. Consider reduced QR–decomposition $A = \hat{Q}\hat{R}$ with $A \in \mathbb{C}^{m \times n}$ and $m \geq n$. Show that $\text{rank}(A) = n$ if and only if $r_{ii} \neq 0$ for all i.

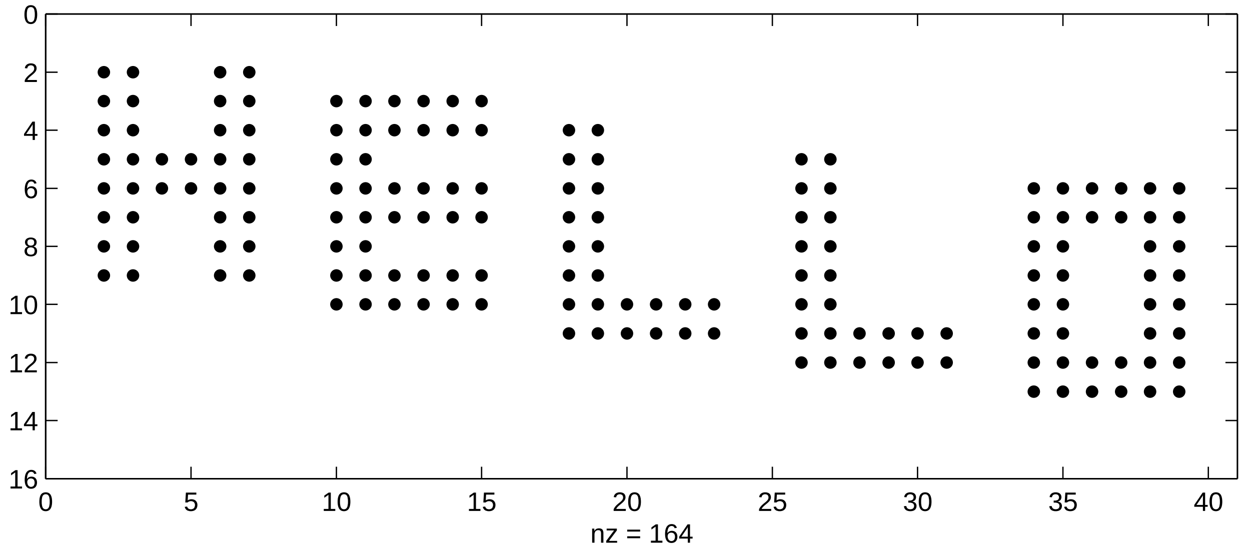
4. Random matrices can be generated with commands `rand` (uniform distribution) and `randn` (normal distribution). Choose some matrices like in exercise 2, and compute the vector v with Matlab. Then choose some random matrices of different size. Check that you always get matrices of full rank. For example you can check that the smallest singular value is not close to zero, so most likely you don't even get matrices which are even close to rank deficient matrices. Try to explain why this is so.

5. T is a Toeplitz matrix if there are numbers b_i such that $t_{ij} = b_{i-j}$. Such matrices can be produced in Matlab with command `toeplitz`. (Toeplitz matrices are really convolution operators, so they appear often in many branches of mathematics and signal processing). Consider a $n \times n$ Toeplitz matrix T_n with $b_0 = 1$, $b_{-1} = 2$ and all other values zero.

   - what are the eigenvalues, determinant and rank of T_n ? Here you don't need Matlab.
   - What is T_n^{-1} ? Experiment with the command `inv` by choosing matrices of different size.
   - Give an upper bound for the least singular value of T_n. (Use the formula for T_n^{-1})
   - Would you say that T_n is "almost singular" for large n ?

Homework

Consider the matrix $A \in \mathbb{R}^{15 \times 40}$ defined as follows: the element $a_{ij} = 1$ if there is a dot in the corresponding place in the picture below and zero otherwise.



Construct the matrix A. The picture was done with *Matlab's* command spy.

- Compute the singular values of A with the command svd. Plot the result with the command plot. What is the rank of A ?

- For $k = 1, ..., \text{rank}(A)$ compute the best rank k approximation to A. Study the quality of the approximation by visualizing these approximations with the command pcolor. (Set first colormap(gray)). You may find the command flipud helpful. What is the smallest k such that the "message" is clearly readable?