Numerical linear algebra autumn 2004

Problem set 4

1. Check the following properties of the condition number:

   – $\kappa(A) \geq 1$ for all A

   – $\kappa(A) = \kappa(cA) = \kappa(A^{-1})$

   – $\kappa_2(A) = \kappa_2(A^*)$

   – U unitary $\implies \kappa_2(U) = 1$

   – U unitary $\implies \kappa_2(A) = \kappa_2(AU) = \kappa_2(UA)$

2. Consider the following matrix

$$A = \begin{pmatrix} 120 & 11 & -0.9 & 0.15 \\ 9.8 & 1.5 & 0.1 & -0.02 \\ -1.1 & 0.2 & -0.03 & 0.001 \\ 0.12 & 0.04 & 0.006 & 0.0004 \end{pmatrix}$$

   Compute the condition number of A with Matlab (using the command `cond`). Try to find some (simple) matrices B_1 and B_2 such that the condition number of B_1AB_2 is much less than the condition number of A.

3. Finding matrices B_1 and/or B_2 as in the previous exercise is called *preconditioning*. Suppose we want to solve the problem $Ax = b$. Explain how to get the solution using the preconditioned matrix $\tilde{A} = B_1AB_2$.

4. In the solution of the least squares problems there appear *Vandermonde* matrices. Matrix V is a Vandermonde matrix if there are numbers a_k such that

$$v_{ij} = a_i^{j-1}$$

   The commands `vander` and `fliplr` will be useful in constructing such matrices. Load the population data: > `load census` . Vector `cdate` contains the years, and vector `pop` the corresponding populations. Plot the data. Find the least squares fit to the data using polynomials of different degree. Which is the lowest degree polynomial which adequately represents the data?

   – Study the conditioning of the system, and in particular check that using normal equations approximately doubles the condition number.

   – Compute the solution of degree 3 using

     → normal equations

     → directly with Matlab's \

     → QR

     → SVD

   Compare the results obtained. Compute also the corresponding residuals.

- You will find that the condition number grows quite fast when the degree of the polynomial grows. You can reduce significantly the condition number using a suitable preconditioning as in exercise 3. Which (diagonal) matrices B_1 and B_2 are useful here?

- Condition number can be reduced more effectively by scaling the problem appropriately. Above you have represented the polynomial as

$$p(t) = \sum_{k=0}^{n} a_k\, t^k$$

However, polynomials can be also represented as

$$p(t) = \sum_{k=0}^{n} a_k \left(\frac{t - t_0}{d}\right)^k$$

where t_0 and d are some convenient parameters. Choose t_0 and d such that the condition number of the resulting Vandermonde matrix is about 10.