Numerical linear algebra autumn 2004

Problem set 5

Oct 12: lecture in M8 at 11 - 13.

Oct 14: exercises in M352 at 14 - 16.

First partial exam: nov 5 in M3 at 12 – 15.

Second partial exam: dec 10 in M3 at 12 – 15.

1. Which of the following statements are true and which are false? Prove the true ones and give counterexamples for the false ones.

   – If λ is an eigenvalue of A and A is nonsingular, then $1/\lambda$ is an eigenvalue of A^{-1}.

   – If A is real and λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

   – If all the eigenvalues of A are zero, then $A = 0$.

   – If $A = A^*$ and λ is an eigenvalue of A, then $|\lambda|$ is a singular value of A.

2. The location of the eigenvalues can be estimated with *Gerschgorin's theorem*. Let A be an arbitrary square matrix and let us define

$$r_i = \sum_{i \neq j} |a_{ij}| \qquad \text{and} \qquad D_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq r_i \}$$

   Then $\Lambda(A) \subset \bigcup_{i=1}^{n} D_i$. The disks D_i are called Gerschgorin's disks. Moreover if k disks form a connected component of the complex plane then this component contains precisely k eigenvalues (counted with algebraic multiplicity). In particular if $D_i \cap D_j = \varnothing$ for all $i \neq j$, then each disk contains one eigenvalue. Use this result to estimate the eigenvalues of the following matrix.

$$A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$$

   Suppose that $|\varepsilon| < 1$. How can you improve the estimate of the smallest eigenvalue to $|\lambda_3 - 1| \leq c\,\varepsilon^2$ where c is some positive constant? Is it possible to have the estimate $|\lambda_3 - 1| \leq c\,\varepsilon^3$?

3. Recall that $\rho(A)$ is the spectral radius. Prove the following.

$$\lim_{k \to \infty} \| A^k \| = 0 \quad \Longleftrightarrow \quad \rho(A) < 1$$

   This can be proved in many ways, but for example Schur decomposition might be useful. Here the norm can be arbitrary. Prove the result first for some convenient norm, and then explain why the result must hold for any norm.

4. How do you produce with Matlab

   – a random symmetric matrix

   – a random unitary matrix

- a random normal matrix
- a random upper triangular matrix
- a random tridiagonal matrix

Note that there may be many reasonable ways to do this.

5. Test the sensitivity of the eigenvalue problem in the following cases with Matlab

   - symmetric matrix
   - defective matrix
   - a "far from normal" matrix. In fact there is no satisfactory measure of nonnormalness. Anyway the idea is to construct a matrix whose eigenvectors are "almost" linearly dependent.

   In each case take some matrices of the above type, perturb them and study the effect on the spectrum.

Homework

Consider random matrices in Matlab: `randn(n)/sqrt(n)` . The square root is included so that the limiting behavior when $n \to \infty$ becomes clearer. Do not try prove anything, but try to justify your answers with relevant arguments and pictures.

- take many random matrices of same size and plot the eigenvalues in the same picture. What appears to be the spectral radius? How does this seem to change when $n = 8$, 16, 32, ...
- how does the 2–norm behave as $n \to \infty$? Of course $\rho(A) \leq \| A \|_2$, but does this inequality seem to approach the equality?
- how about condition number in 2–norm. Given n, what proportion of random matrices seem to have $\kappa_2(A) \leq 1/2, 1/4, 1/8, \ldots$? How does this change when $n \to \infty$?