Numerical linear algebra autumn 2004

Problem set 6

1. The solution to the linear system $Ux = b$ where U is an upper triangular matrix can be computed with *back substitution*. The general step in this is

$$u_{ii}x_i = b_i - u_{i,i+1}x_{i+1} - \ldots - u_{i,n}x_n$$

Show that the total number of the flops required is n^2.

2. Let us consider a 3×3 matrix A. We want to introduce zeros by left- and/or right-multiplication by orthogonal matrices Q_j like Householder reflectors. Consider the following structures:

$$(i) \begin{pmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix} \qquad (ii) \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix} \qquad (iii) \begin{pmatrix} \times & \times & 0 \\ 0 & 0 & \times \\ 0 & 0 & \times \end{pmatrix}$$

Here $\times$ denotes some nonzero element. For each structure which of the following situation holds:

I. Can be obtained by a sequence of left-multiplications by matrices Q_j

II. Not case I, but can be obtained by left- and right-multiplication by matrices Q_j

III. Cannot be obtained by any sequence of left- and right-multiplication by matrices Q_j

3. If we can find the inverse matrix rapidly then we can also multiply matrices rapidly. More precisely show that

   – suppose that we can compute the inverse with $O(n^\alpha)$ flops; then we can compute the product with $O(n^\alpha)$ flops.

Hint: consider the following matrix:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

4. Let us consider the following matrix

$$G = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \qquad \det(G) = 1$$

This is called *Givens rotation*. (Why rotation? What is the angle of rotation?) Then let A be a square matrix of size $n \times n$. Multiplying A from the left by a suitable Householder reflector H gives us

$$HA = \begin{pmatrix} \times & \times & \ldots & \times \\ 0 & \times & \ldots & \times \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \times & \ldots & \times \end{pmatrix}$$

Show that the same effect can be obtained using $n-1$ Givens rotations. Which method is more efficient?

5. Consider the polynomial $p(z) = c_0 + c_1 z + ... + c_k z^k$. For a square matrix we can in the same way define $p(A) = c_0 + c_1 A + ... + c_k A^k$. How many flops is needed to evaluate $p(z)$ by the obvious method? How many flops is needed using the *Horner's rule*:

$$p(z) = c_0 + z\,(c_1 + z\,(c_2 + ... + z\,(c_{k-1} + c_k\,z)...))$$

   How many flops is needed to evaluate $p(A)$? Show that now the Horner's rule does not really help. Why?

Homework

Program Strassen's algorithm with Matlab. Consider only the case where the matrices are of size $2^k \times 2^k$. Compare the performance with the standard algorithm. The time required can be measured with tic, toc, etime or cputime.