Numerical linear algebra autumn 2004

Problem set 7

1. Let $A \in \mathbb{R}^{n \times n}$ be nonsymmetric with $Aq = \lambda q$ and let x be an approximation to the eigenvector q. Show that in this case the Rayleigh quotient gives only a linear approximation to the eigenvalue, in other words

$$|r(x) - \lambda| = O(|x - q|)$$

2. In inverse iteration and Rayleigh iteration one has to solve an ill–conditioned system

$$(A - \mu I)w = v$$

   where μ is an approximate eigenvalue. Why is this not a problem in practice?

   Hint: without loss of generality consider the case where A has one very small eigenvalue compared to others. In this case then $\mu = 0$. Then think about v as

$$v = c_1 q^1 + \ldots + c_n q^n$$

   where q^i are the eigenvectors of A. Now let $\tilde{w}$ be the computed solution and w the exact solution. Why is $|\tilde{w}/|\tilde{w}| - w/|w||$ in general small although $|\tilde{w} - w|$ is big?

3. Let A be a square matrix in Hessenberg form and let $A = QR$ be its QR–decomposition. Show that Q and RQ are also Hessenberg matrices. What if A is tridiagonal? Are Q and RQ also tridiagonal?

   Check this with Matlab. The Hessenberg form of any matrix can be computed with command `hess`.

4. Take some matrices, symmetric and unsymmetric, and compute some eigenvalues with Rayleigh iteration. Does the symmetric case seem to be significantly faster? Are the computations significantly faster, if you first reduce to tridiagonal/Hessenberg form? How would you compute a complex eigenvalue? You can compare your results with the result given by the command `eig`.

5. Compute the time required for QR–decomposition for general matrices and Hessenberg matrices of different size. Is the QR–decompostion for Hessenberg matrices significantly faster than QR–decomposition for general matrices?