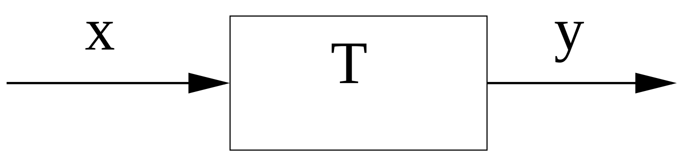
Wavelets, spring 2002

1 Basic facts about filters

Here we simply recall some convenient terminology. A *digital signal* is a sequence $x = (\dots, x_{-1}, x_0, x_1, \dots)$, in other words a map $x : \mathbb{Z} \to \mathbb{C}$. Sometimes it is also useful to consider signal as a sum of Diracs:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \delta(t-k)$$

A *filter* (or system) is then a function which maps one sequence (input) to another (output or response).



Kuva 1.1: System/filter T with one input x and one output y.

System T is *timeinvariant*, if for all k

$$TS_k = S_k T$$

where S_k is a shift. The timeinvariant systems can be characterized as follows:

Theorem 1.1 *T is timeinvariant if and only if there is a sequence t such that*

$$y = Tx \quad \Leftrightarrow \quad y_n = \sum_{k=-\infty}^{\infty} t_k x_{n-k}$$

The above sum is called *convolution* of sequences t and x, and it is denoted by $y = t * x$. So although signals and filters are physically completely different kind of objects, they are mathematically both represented by sequences.
Let $x_n = e^{i\omega n}$. Then

$$y_n = \sum_{k=-\infty}^{\infty} t_k x_{n-k} = \sum_{k=-\infty}^{\infty} t_k e^{i\omega(n-k)} = \Big(\sum_{k=-\infty}^{\infty} t_k e^{-i\omega k}\Big) e^{i\omega n}$$

The function

$$T(\omega) = \sum_{k=-\infty}^{\infty} t_k e^{-i\,\omega k}$$

is called the *transfer function* of the system. On the other hand if t represents a signal, then we say that $T(\omega)$ is the Fourier transform of t. This is justified by considering t as a sum of Diracs. Note finally that $T(\omega)$ is 2π–periodic. The connection between transfer functions and convolution is:

$$y = t * x \quad \Leftrightarrow \quad Y(\omega) = T(\omega)X(\omega)$$

Sometimes it is convenient to put $z = e^{i\,\omega}$ and define

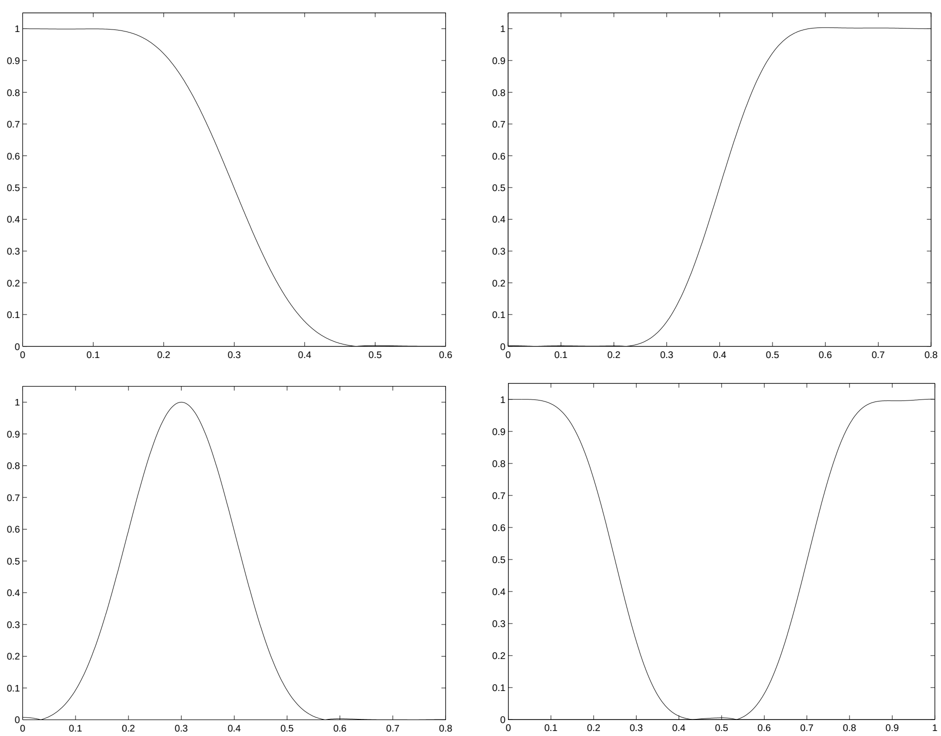
$$T(z) = \sum_{k=-\infty}^{\infty} t_k z^{-k}$$

This is the $\mathcal{Z}$ *transform* of the system.
We may write $T(\omega)$ as follows:

$$T(\omega) = |T(\omega)| e^{i\,\varphi(\omega)}$$

where $|T(\omega)|$ is *amplification or gain* and $\varphi(\omega)$ is *phase*. The following terminology is convenient:

- low pass filter: $|T(\omega)| \approx 1$, when $\omega < \omega_0$ and $|T(\omega)| \approx 0$ when $\omega > \omega_0$ for some ω_0.
- high pass filter: $|T(\omega)| \approx 0$, when $\omega < \omega_0$ and $|T(\omega)| \approx 1$ when $\omega > \omega_0$.
- band pass filter: $|T(\omega)| \approx 1$, when $\omega_1 < \omega < \omega_2$ and $|T(\omega)| \approx 0$ when $\omega < \omega_1$ or $\omega > \omega_2$.
- band stop filter: $|T(\omega)| \approx 0$, when $\omega_1 < \omega < \omega_2$ and $|T(\omega)| \approx 1$ when $\omega < \omega_1$ or $\omega > \omega_2$.



Kuva 1.2: Examples of filters: low pass, high pass, band pass and band stop