Wavelets, spring 2002

Problem set 1

Let us recall the following *vector spaces*:

$$L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(t)|dt < \infty\}$$
$$L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\}$$

The corresponding *norms* are:

$$||f||_1 = \int_{-\infty}^{\infty} |f(t)|dt$$
$$||f||_2 = \Big(\int_{-\infty}^{\infty} |f(t)|^2 dt\Big)^{1/2}$$

In $L^2(\mathbb{R})$ we have also an *inner product*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$$

Functions f and g are *orthogonal* if $\langle f, g\rangle = 0$.
The Fourier transform and the inverse Fourier transform are defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega$$

We will often need the following

Theorem 0.1

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle$$

In particular this implies that

$$||f||_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \frac{1}{2\pi}||\hat{f}||_2^2$$

The latter result is usually known as **Parseval formula**. Convolution is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)dt$$

On can show that the following properties are true:

$$\begin{aligned} & f * g = g * f \\ & \widehat{f * g} = \hat{f}\,\hat{g} \\ & ||f * g||_1 \leq ||f||_1 ||g||_1 \end{aligned}$$

1. Give an example of a function such that

$$\begin{array}{llll} (1) & f \in L^2(\mathbb{R}) & \text{and} & f \notin L^1(\mathbb{R}) \\ (2) & f \in L^1(\mathbb{R}) & \text{and} & f \notin L^2(\mathbb{R}) \end{array}$$

2. Give an example of functions f and g such that $f \notin L^1(\mathbb{R})$ and $g \notin L^1(\mathbb{R})$, but $(f * g)(t)$ is well defined for all t.

3. Let $p(t) = a_0 + a_1 t + \cdots + a_k t^k$, $q(t) = b_0 + b_1 t + \cdots + b_n t^n$ and $f(t) = p(t)/q(t)$. Let us suppose that q doesn't have any real roots. Show that

$$\begin{aligned} k < n \quad & \Rightarrow \quad f \in L^2(\mathbb{R}) \\ k < n-1 \quad & \Rightarrow \quad f \in L^1(\mathbb{R}) \end{aligned}$$

4. Let $S \subset L^2(\mathbb{R})$, and let us define

$$S^{\perp} = \{ f \in L^2(\mathbb{R}) \,|\, \langle f, g \rangle = 0 \ , \ g \in S \}$$

   Show that $S^{\perp}$ is a vector subspace of L^2.

5. Use Fourier transform and convolution to show that

$$\int_{-\infty}^{\infty} \frac{\sin^3(t)}{t^3} dt = \frac{3\pi}{4}$$

6. Compute the Fourier transforms of $f_1(t) = \sin(t)$ and $f_2(t) = \cos(t)$.

7. Let

$$g(t) = \begin{cases} 1 - |t| \ , & |t| \leq 1 \\ 0 \ , & otherwise \end{cases}$$

   and let $g_{b,\omega}(t) = g(t-b)e^{i\omega t}$. Compute the sides and the area of the Heisenberg box associated to atoms $g_{b,\omega}$. How many percent is the area bigger than the minimum area?