Wavelets, spring 2002

Problem set 3

1. Choose a polynomial p such that $\psi(t) = p(t)e^{-t^2}$ is a wavelet with 3 vanishing moments. Is ψ rapidly decreasing? Is $\psi \in \mathcal{S}(\mathbb{R})$? This problem is most conveniently done with *Maple*.

2. Let $u = (1, 0)$, $v = (0, 1)$ and $w = (1, 1)$. Show that $\{u, v, w\}$ is a frame in $\mathbb{R}^2$ and compute the frame bounds.

3. Consider the Haar wavelet system:

$$\psi(t) = \begin{cases} 1 \, , & 0 < t < 1/2 \\ -1 \, , & 1/2 < t < 1 \\ 0 \, , & otherwise \end{cases} \qquad \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

   Then given some $f \in L^2(\mathbb{R})$ we may compute the wavelet coefficients $c_{j,k} = \langle f, \psi_{j,k} \rangle$, and we have the reconstruction formula

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(t)$$

   - show that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal set
   - show that
$$||f||_2^2 = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{j,k}|^2$$
   - let us define f by
$$f(t) = \begin{cases} 2t \, , & 0 < t \leq 1 \\ 2 \, , & 1 < t \leq 4 \\ 6 - t \, , & 4 < t < 6 \\ 0 \, , & otherwise \end{cases}$$

     Compute the wavelet coefficients of f. How do the coefficients behave when $j \to \infty$? How do the coefficients behave when $j \to -\infty$? To compress the signal one discards all the coefficients that are smaller that 10^{-5}. How many coefficients remain? You don't need to compute the exact number, but just to give a reasonble upper bound.

This is for those who like to program. The previous problem can be done algorithimcally:

INPUT: function f and its support $[t_0, t_1]$, tolerance ε
OUTPUT: the wavelet coefficients which are bigger than tolerance

To proceed, do the following:

- first estimate what is the coarsest level you need. In other words we need j_0 such that $|c_{j,k}| < \varepsilon$ for all $j < j_0$.
- Then starting with level j_0, compute the relevant coefficients. The program might look like this:

  j:=j0
  repeat
  compute k_0 and k_1 such that $c_{j,k}$ can be nonzero only for $k_0 \leq k \leq k_1$
  compute $c_{j,k}$ for $k_0 \leq k \leq k_1$
  discard values for which $|c_{j,k}| < \varepsilon$
  let n be the number of coefficients that remain
  j:=j+1
  until $n = 0$

- Think about the stopping criterium. Does it always give reasonable results? What other criteria could be used? Take different functions and examine how much compression you get with different tolerances.