Wavelets, spring 2002

Problem set 4

1. Let φ be a scaling function. Show that

$$\sum_{k=-\infty}^{\infty} \varphi(t-k) = 1$$

for all t. Hint: the left hand side is a periodic function. What is its Fourier series?

2. Let ψ be a wavelet associated to MRA. Show that $\hat{\psi}(4\pi n) = 0$ for all n.

3. **Shannon wavelets** Let us start with the following subspaces of $L^2(\mathbb{R})$:

$$V_j = \{f \in L^2(\mathbb{R}) \,|\, \mathrm{supp}(\hat{f}) \subset [-2^j\pi, 2^j\pi]\}$$

With this choice, the properties (1), (3) and (4) of MRA are satisfied. But what is the corresponding φ? Let us define

$$\varphi(t) = \frac{\sin(\pi t)}{\pi t} \quad \Leftrightarrow \quad \hat{\varphi}(\omega) = \begin{cases} 1\ , & |\omega| < \pi \\ 0\ , & otherwise \end{cases}$$

The Shannon sampling theorem says that the translates of φ really span the whole V_0. In other words a band limited signal can be reconstructed from its samples.

- Check that the translates $\varphi(t-k)$ are really orthogonal to each other
- compute the coefficients h_k in the scaling equation

$$\varphi(t) = 2\sum_{k=-\infty}^{\infty} h_k \varphi(2t-k)$$

- what is the associated wavelet?

Hint: in all cases work in the Fourier domain.

Shannon wavelets are then well localised in frequency, but badly localised in time, so they are in a sense "opposite" to Haar system.

4. Try to find some appropriate h and then compute the scaling functions and wavelets with Matlab program `skaalaf.m` Also you get the corresponding transfer functions with `transfer.m`

   For example with 4 parameters we have the equations

$$
\begin{aligned}
h_0 + h_2 &= 1/2 \\
h_1 + h_3 &= 1/2 \\
h_2 &= \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{1 + 8h_3 - 16h_3^2}
\end{aligned}
$$

   So there are real solutions if

$$-0.1036 \approx \tfrac{1}{4} - \tfrac{\sqrt{2}}{4} \le h_3 \le \tfrac{1}{4} + \tfrac{\sqrt{2}}{4} \approx 0.6036$$

   Try some different parameters and plot some pictures of the functions.