Wavelets, spring 2002

1 Some function spaces

Let V be a vector space and let $n : V \to \mathbb{R}$. The map n is a *norm* if the following conditions are satisfied:

1. $n(x) \geq 0$ for all $x \in V$.
2. $n(cx) = |c|n(x)$ for all $x \in V$ and all $c \in \mathbb{C}$.
3. $n(x + y) \leq n(x) + n(y)$ for all $x, y \in V$. This is called the *triangle inequality.*
4. $n(x) = 0$ only if $x = 0$.

For example if $V = \mathbb{C}^n$, then $n(x) = \max_{1 \leq i \leq n} |x_i|$ is a norm.
Let us consider the following vector spaces.

Definition 1.1

$$C^n(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, f,\ f', \ldots, f^{(n)} \textit{ continuous} \right\}$$

If all derivatives of f are continuous we say that f is smooth and the space of smooth functions is denoted by $C^\infty(\mathbb{R})$.

The bigger the parameter n, the more regular f is. Of course instead of $C^n(\mathbb{R})$ we may consider the space $C^n([t_0, t_1])$ where functions are defined or analysed only in the interval $[t_0, t_1]$.
So a statement like $f \in C^n(\mathbb{R})$ for some n gives information about the *global* regularity of the signal. We would like to measure also *local* regularity, and also to refine the notion of regularity.

Example 1.1 *Let $f(t) = \sqrt{t}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Now obviously $f \in C^0(\mathbb{R})$, but $f \notin C^1(\mathbb{R})$ because the derivative is not continuous at the origin. However, f is anyway more than just continuous at the origin.*

Let us first first give a pointwise definition.

Definition 1.2 *Let $0 < \alpha < 1$; we say that f is α–Lipschitz at point t_0 if there is a constant c such that*

$$|f(t) - f(t_0)| \leq c|t - t_0|^\alpha$$

for all t sufficiently close to t_0. Let n be an integer and let f have n continuous derivatives at t_0. We say that f is $n+\alpha$–Lipschitz at point t_0 if there is a constant c such that

$$|f^{(n)}(t) - f^{(n)}(t_0)| \leq c|t - t_0|^\alpha$$

We may also say that f has a Lipschitz number $n+\alpha$ at t_0.

The function in the previous example is evidently 1/2–Lipschitz and $f(t) = |t|^{3/2}$ is 3/2–Lipschitz at the origin.[1]

Definition 1.3 *Let $0 < \alpha < 1$ and put*

$$C^\alpha(\mathbb{R}) = \big\{ f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ is } \alpha\text{–Lipschitz for all } t \big\}$$
$$C^{n+\alpha}(\mathbb{R}) = \big\{ f : \mathbb{R} \to \mathbb{C} \,|\, f \in C^n(\mathbb{R}) \text{ and } f^{(n)} \in C^\alpha(\mathbb{R}) \big\}$$

2 Continuous wavelet transform

Definition 2.1 *A* wavelet *is a function $\psi : \mathbb{R} \to \mathbb{R}$ such that*

(i) $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

(ii) $\int_{-\infty}^{\infty} \psi(t)dt = 0$

This implies that $\hat{\psi}$ (the Fourier transform of ψ) is continuous, and that $\hat{\psi}(0) = 0$, and hence

$$c_\psi = \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

Recall that a signal, for purposes of this course, is a function in $L^2(\mathbb{R})$.

Definition 2.2 *A (continuous) wavelet transform of a signal f is*

$$W_f(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)dt$$

where $a > 0$, $b \in \mathbb{R}$, $\psi_{a,b}(t) = a^{-1/2}\psi\big((t-b)/a\big)$ and ψ is a wavelet.

[1] Defining Lipschitz regularity for integer values makes also sense. The resulting function spaces are called *Zygmund classes*. However, this leads to some technical complications which we prefer to avoid.

Note that the word continuous doesn't imply that any of the functions were continuous. It refers to the fact that the parameters a and b vary continuously. One easily checks that $||\psi_{a,b}||_2 = ||\psi||_2$. Then by Cauchy inequality we get

$$|W_f(a,b)| = |\langle f, \psi_{a,b}\rangle| \leq ||f||_2||\psi_{a,b}||_2 = ||f||_2||\psi||_2$$

Hence W_f is a bounded function.

Theorem 2.1 *If ψ is a wavelet, then we have the reconstruction formula:*

$$f(t) = \frac{1}{c_\psi}\int_0^\infty \int_{-\infty}^\infty W_f(a,b)\psi_{a,b}(t)\frac{da\,db}{a^2}$$

and the energy formula:

$$c_\psi \int_{-\infty}^\infty |f(t)|^2 dt = \int_0^\infty \int_{-\infty}^\infty |W_f(a,b)|^2\frac{da\,db}{a^2}$$