Wavelets, spring 2002

3 Regularity and singularity

To be able to effectively analyse the singularities of the function/signal we need to put some more conditions on the wavelet.

Definition 3.1 *A function f is said to be* rapidly decreasing *(or to have fast decay) if for all n there are positive constants C_n such that*

$$|f(t)| \leq \frac{C_n}{1 + |t|^n}$$

The space of functions which are in $C^\infty(\mathbb{R})$ and whose all derivatives are rapidly decreasing is denoted by $\mathcal{S}(\mathbb{R})$.

For example all continuous functions with compact support are rapidly decrasing. Also $f_1(t) = e^{-|t|}$ and $f_2(t) = e^{-t^2}$ are rapidly decrasing. In addition $f_2 \in \mathcal{S}(\mathbb{R})$.

Definition 3.2 *A function f is said to have n vanishing moments if*

$$\int_{-\infty}^{\infty} t^k f(t)dt = 0 \qquad 0 \leq k < n$$

Note that a wavelet has always at least one vanishing moment. Note also that by properties of the Fourier transform this is equivalent to

$$\hat{f}^{(k)}(0) = 0 \qquad 0 \leq k < n$$

Hence one way to produce functions with vanishing moments is to start with $\hat{f}(\omega) = \omega^n g(\omega)$ where g is some suitable function, and then computing the inverse Fourier transform of $\hat{f}$.
To measure the regularity of the signal we need wavelets with vanishing moments. Let us also recall that a signal is always a function in $L^2(\mathbb{R})$. So let us suppose that ψ is a wavelet such that

- $\psi \in C^n(\mathbb{R})$
- ψ has n vanishing moments
- ψ and its derivatives up to order n are rapidly decreasing

Further let k be an integer, $0 < \alpha < 1$ and $k + \alpha < n$. Then we have the following theorem

Theorem 3.1 *If a signal f is $k + \alpha$–Lipschitz in the interval $[t_0, t_1]$, then there is a constant C such that*

$$|W_f(a, b)| \leq C a^{k+\alpha+1/2} \qquad \text{for all} \quad t_0 \leq b \leq t_1 \quad \text{and} \quad a > 0$$

Conversely if f is bounded and the above inequality is satisfied, then f is $k + \alpha$–Lipschitz in the interval $[t_0 + \varepsilon, t_1 - \varepsilon]$ for all $\varepsilon > 0$.

So we see that for the purposes of analysis it would nice to have wavelets many vanishing moments and many continuous derivatives. On the other hand it is most convenient to work with wavelets with compact support, in other words timelimited wavelets. [1]. The following result indicates some limitations.

Theorem 3.2 *If the wavelet ψ has a compact support, then all its moments cannot vanish. On the other hand there is a wavelet $\psi \in \mathcal{S}(\mathbb{R})$ such that all its moments vanish.*

4 Some preparations

Before we can start discussing *discrete wavelet transform*, let us review some basic notions.

Definition 4.1 *Let $c = (\ldots, c_{-1}, c_0, c_1, c_2, \ldots)$. Then c belongs to the space $l^2(\mathbb{Z})$ if*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$$

The norm is $||c||_2 = \left(\sum_{-\infty}^{\infty} |c_k|^2\right)^{1/2}$ and the inner product is $\langle c, d\rangle = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k$. c belongs to the space $l^1(\mathbb{Z})$ if

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty$$

The norm is $||c||_1 = \sum_{-\infty}^{\infty} |c_k|$.

[1] Recall that in signal processing literature a signal with compact support is called time limited, and a signal whose Fourier transform has compact support is called band limited

So both $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ are inner product spaces, and hence in both cases we say that two elements are orthogonal if their inner product is zero. In fact they are both *Hilbert spaces*.

Definition 4.2 *Let V be a normed vector space and f_j a sequence of elements of V. We say that f_j converges to an element of $f \in V$ if*

$$||f - f_j|| \to 0 \quad \textit{when} \quad j \to \infty$$

f_j is called a Cauchy sequence *if for any $\varepsilon > 0$ there is n such that*

$$||f_k - f_j|| < \varepsilon \quad \textit{when} \quad j\,, k > n$$

If all Cauchy sequences converge, the space is said to be complete*.*

It is rather straightforward to show that if f_j converges it must be a Cauchy sequence. However, the converse is not always true.
One usually says that a complete normed space is a *Banach space* and a complete inner product space is a *Hilbert space*. Recall that inner product space is a normed space where the norm is given by inner product: $||f|| = \sqrt{\langle f, f\rangle}$.

Theorem 4.1 *The spaces are $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^1(\mathbb{R})$ and $l^1(\mathbb{Z})$ are complete.*

In signal analysis one wants to represent or decompose a signal using some "atoms" or basis functions. A familiar example is Fourier series where the basis functions are sines and cosines. More generally for a given signal f we may try to represent it as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi_k(t)$$

where φ_k are given functions and the constants c_k are to be determined. In the follwing we will for simplicty consider only Hilbert spaces. Recall that in continuous wavelet transform we get the information about the signal by evaluating inner products. Similarly in the discrete case let us consider a collection $\{\varphi_k\} \in V$ where V is a Hilbert space. Then for any $f \in V$ we can compute the coefficients $c_k = \langle f, \varphi_k\rangle$. But how useful the conefficients c_k are? The following definition has been found appropriate.

Definition 4.3 *Let V be a Hilbert space. A collection $\{\varphi_k\}$ is called a* frame *if there are positive constants A and B such that*

$$A||f||^2 \le \sum_{k=-\infty}^{\infty} |\langle f, \varphi_k\rangle|^2 \le B||f||^2$$

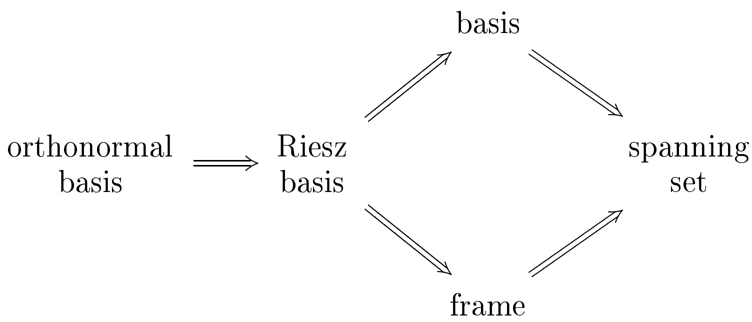
The constants A and B are called *frame bounds* and the frame is *tight* if $A = B$.
We may interprete the above inequalities as follows. Let $c_k = \langle f, \varphi_k \rangle$. Then we can write $A||f||_2^2 \leq ||c||_2^2 \leq B||f||_2^2$, which implies that c cannot big is f is small ($B < \infty$) and on the other hand c cannot be small if f is big ($A > 0$).[2]
Note that the frame is not necessarily linearly independent. A closely related notion is a *basis*.

Definition 4.4 *Let V be a Hilbert space. A collection $\{\varphi_k\}$ is called a* basis *if for all $f \in V$ there are unique constants c_k such that*

$$f = \sum_{k=-\infty}^{\infty} c_k \varphi_k$$

A basis is which is also a frame is called Riesz basis. *Finally the basis is* orthogonal *if $\langle \varphi_n, \varphi_k \rangle = 0$ for $n \neq k$ and* orthonormal *if in addition $||\varphi_k|| = 1$ for all k.*

Now one can show that *if the frame is linearly independent, then it is a Riesz basis.* Hence we have the following implications:



5 Discrete wavelet transform

Let us define

$$c_{j,k} = W_f(2^{-j}, 2^{-j}k) = \langle f, \psi_{j,k} \rangle$$

where $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ and $j,\, k \in \mathbb{Z}$.
A minimal requirement for the wavelet is the following. Suppose that $c_{j,k}$ are wavelet coefficients of f and $d_{j,k}$ are wavelet coefficients of g. We want that the coefficients characterize the signal completely in the sense that if

[2]We will not use frames in the following, but they are quite important in some applications of wavelets.

$c_{j,k} = d_{j,k}$ for all j and k then it must follow that $f = g$. This is the same as requiring that the set $\{\psi_{j,k}\}$ spans $L^2(\mathbb{R})$. However, for practical purposes we must also require that the set $\{\psi_{j,k}\}$ is a *frame*. This is because finally in practise all computations must be done numerically, and the frame bounds guarantee that the computations are numerically stable.
However, since we will not consider frames anymore, we will directly take up the case where the set $\{\psi_{j,k}\}$ is linearly independent; in other words

the set $\{\psi_{j,k}\}$ should be a Riesz basis

The best case is evidently:

the set $\{\psi_{j,k}\}$ is an orthonormal basis

Now it's not obvious how to find ψ such that $\{\psi_{j,k}\}$ is an orthonormal basis. In fact it's not so evident if such ψ exists at all. However, it turns out that one can produce such ψ's with the help of *multiresolution analysis*.