Wavelets, spring 2002

6 Multiresolution analysis

By *multiresolution analysis* (MRA) we mean the following:

Definition 6.1. *MRA in $L^2(\mathbb{R})$ is a collection of subspaces $V_j \subset L^2(\mathbb{R})$ and a* scaling function *$\varphi \in L^2(\mathbb{R})$ such that*

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots$

(2) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$

(3) $\cap_{j\in\mathbb{Z}} V_j = 0$

(4) $closure\big(\cup_{j\in\mathbb{Z}} V_j\big) = L^2(\mathbb{R})$

(5) $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis of V_0.*

Let us further set $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k)$. One can then immediately check that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j.
Then we can define a sequence of space W_j as follows: W_j is the orthogonal complement of V_j in V_{j+1}. We denote this by $V_{j+1} = V_j \oplus W_j$.

Definition 6.2. *A* wavelet *ψ associated to MRA is a function such that $\{\psi(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0.*

Again we set $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ and check that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j. Let us also recall that we will always suppose that

$$\begin{aligned}
\varphi \in & L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad \text{and} \quad \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(t)dt = 1 \\
\psi \in & L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad \text{and} \quad \hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)dt = 0
\end{aligned}$$

This implies in particular that the Fourier transforms $\hat{\varphi}$ and $\hat{\psi}$ are continuous. Now we have defined everything in terms of subspaces V_j and W_j but at this point it's not clear if there really exist some functions φ and ψ which satisfy the given conditions. So how could we find such functions?

By property (2) and (5) of Definition 6.1 there must be constants h_k such that

$$\tfrac{1}{2}\varphi(t/2) = \sum_{k=-\infty}^{\infty} h_k \varphi(t-k) \tag{6.1}$$

Taking Fourier transforms we get

$$\hat{\varphi}(2\omega) = \sum_{k=-\infty}^{\infty} h_k \hat{\varphi}(\omega) e^{-ik\omega} = \Big(\sum_{k=-\infty}^{\infty} h_k e^{-ik\omega}\Big)\hat{\varphi}(\omega) = m_0(\omega)\hat{\varphi}(\omega) \tag{6.2}$$

Since $\hat{\varphi}(0) \neq 0$ this implies that $m_0(0) = \sum_{k=-\infty}^{\infty} h_k = 1$. Similarly there must be constants g_k such that

$$\tfrac{1}{2}\psi(t/2) = \sum_{k=-\infty}^{\infty} g_k \varphi(t-k) \tag{6.3}$$

Again taking Fourier transforms we get

$$\hat{\psi}(2\omega) = \Big(\sum_{k=-\infty}^{\infty} g_k e^{-ik\omega}\Big)\hat{\varphi}(\omega) = m_1(\omega)\hat{\varphi}(\omega) \tag{6.4}$$

Since $\hat{\psi}(0) = 0$ this implies that $m_1(0) = \sum_{k=-\infty}^{\infty} g_k = 0$. In the following we will always suppose that

$$\begin{aligned} h \in & l^1(\mathbb{Z}) \cap l^2(\mathbb{Z}) \quad \text{and} \quad m_0(0) = \sum_{k=-\infty}^{\infty} h_k = 1 \\ g \in & l^1(\mathbb{Z}) \cap l^2(\mathbb{Z}) \quad \text{and} \quad m_1(0) = \sum_{k=-\infty}^{\infty} g_k = 0 \end{aligned} \tag{6.5}$$

From this it follows that m_0 and m_1 are continuous.

Definition 6.3. *h is the* (low pass) filter *associated to φ and m_0 is the corresponding* transfer function. *Similarly g is the* (high pass) filter *associated to ψ and m_1 is its* transfer function.

Now we can apply the equation (6.2) iteratively:

$$\hat{\varphi}(\omega) = m_0(\omega/2)\hat{\varphi}(\omega/2) = m_0(\omega/2)m_0(\omega/4)\hat{\varphi}(\omega/4) = \hat{\varphi}(2^{-k}\omega)\prod_{j=1}^{k} m_0(2^{-j}\omega)$$

But since $\hat{\varphi}$ and m_0 are continuous, and $\hat{\varphi}(0) = m_0(0) = 1$, we can hope that it's possible to consider the limit $k \to \infty$ and *define*

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega) \tag{6.6}$$

Hence we might proceed as follows:

- choose an appropriate transfer function m_0
- compute $\hat{\varphi}$ by (6.6)
- compute φ by taking inverse Fourier transform of $\hat{\varphi}$
- choose an appropriate filter g
- compute ψ by (6.3)

We could also do:

- choose an appropriate filter h
- compute φ by solving iteratively (6.1)
- choose an appropriate filter g
- compute ψ by (6.3)

But how to choose filters h and g, or equivalently the corresponding transfer functions m_0 and m_1? It is clear that if we just have the conditions (6.5) and follow the procedure above, the result will not be MRA.

6.1 φ, h and m_0

Definition 6.4. *A shift S_n on sequences is defined by*

$$y = S_n x \quad \Leftrightarrow \quad y_k = x_{k-n}$$

So if $n > 0$ (respectively $n < 0$), x is shifted to the right or delayed (resp. to the left).
Let us then suppose that φ verifies all the properties listed in Definition 6.1. What does this imply about h and m_0?

Lemma 6.1. *Let φ be a scaling function and define*

$$a(\omega) = \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi n)|^2$$

Then $a(\omega) = 1$ for all ω.

Proof. Note first that a is 2π–periodic. Hence it has a Fourier series $a(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega}$ where

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_0^{2\pi} a(\omega)e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi n)|^2 e^{-ik\omega}d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} |\hat{\varphi}(\omega + 2\pi n)|^2 e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{-ik\omega}d\omega \end{aligned}$$

But by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)e^{-ik\omega}\overline{\hat{\varphi}(\omega)}d\omega = \int_{-\infty}^{\infty} \varphi(t-k)\varphi(t)dt = 0$$

where the final equality follows from the orthogonality of the translates of φ (property (5) in Definition 6.1). □

With this technical lemma we get

Theorem 6.1. *If φ is a scaling function, then*

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$$

In particular we see that $m_0(\pi) = 0$, so that it is really a low pass filter at least in a weak sense, and $|m_0(\omega)| \leq 1$ for all ω.

Proof.

$$\begin{aligned} 1 = a(2\omega) &= \sum_{n=-\infty}^{\infty} |\hat{\varphi}(2\omega + 2\pi n)|^2 = \sum_{n=-\infty}^{\infty} |m_0(\omega + \pi n)|^2 |\hat{\varphi}(\omega + \pi n)|^2 \\ &= \sum_{n=-\infty}^{\infty} |m_0(\omega + 2\pi n)|^2 |\hat{\varphi}(\omega + 2\pi n)|^2 + \sum_{n=-\infty}^{\infty} |m_0(\omega + \pi + 2\pi n)|^2 |\hat{\varphi}(\omega + \pi + 2\pi n)|^2 \\ &= |m_0(\omega)|^2 \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi n)|^2 + |m_0(\omega + \pi)|^2 \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + \pi + 2\pi n)|^2 \\ &= |m_0(\omega)|^2 a(\omega) + |m_0(\omega + \pi)|^2 a(\omega + \pi) = |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 \end{aligned}$$

□

Theorem 6.2. *If φ is a scaling function, then*

$$\langle h, S_{2n}h\rangle = \begin{cases} 1/2\ , & n = 0 \\ 0\ , & n \neq 0 \end{cases}$$

In particular $\{S_{2n}h\}_{n\in\mathbb{Z}}$ is an orthogonal set in $l^2(\mathbb{Z})$.

Proof.

$$\varphi(t) = 2\sum_{k=-\infty}^{\infty} h_k\varphi(2t-k)$$

$$\varphi(t-n) = 2\sum_{k=-\infty}^{\infty} h_k\varphi(2t-2n-k) = 2\sum_{k=-\infty}^{\infty} h_{k-2n}\varphi(2t-k)$$

Hence

$$\langle\varphi(t), \varphi(t-n)\rangle = 4\langle\sum_{j=-\infty}^{\infty} h_j\varphi(2t-j), \sum_{k=-\infty}^{\infty} h_{k-2n}\varphi(2t-k)\rangle =$$

$$4\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_j h_{k-2n}\langle\varphi(2t-j), \varphi(2t-k)\rangle = 2\sum_{k=-\infty}^{\infty} h_k h_{k-2n} = 2\langle h, S_{2n}h\rangle$$

The result then again follows from property (5) in Definition 6.1. □

6.2 ψ, g and m_1

We can now proceed similarly as above. Note that the essential property we used was the orthogonality of the translates $\varphi(t-k)$. But if ψ is the associated wavelet (Definition 6.2), then its translates are also orthogonal. Hence we can immediately state

Theorem 6.3. *If ψ is a wavelet associated to a MRA, then*

$$\langle g, S_{2n}g\rangle = \begin{cases} 1/2\ , & n = 0 \\ 0\ , & n \neq 0 \end{cases}$$

$$|m_1(\omega)|^2 + |m_1(\omega+\pi)|^2 = 1$$

So at this point the only difference between h and g is that

$$m_0(0) = \sum_{k=-\infty}^{\infty} h_k = 1\ = \sum_{k=-\infty}^{\infty} (-1)^k g_k = m_1(\pi)$$

$$m_0(\pi) = \sum_{k=-\infty}^{\infty} (-1)^k h_k = 0\ = \sum_{k=-\infty}^{\infty} g_k = m_1(0)$$

So h must be a low pass filter and g must be a high pass filter. But there must be much closer connection between wavelets and scaling functions and the corresponding filters.

6.3 Connection

We haven't yet used the fact that $\varphi(t-k)$ and $\psi(t-n)$ should be orthogonal to each other for any k and n. Of course it's sufficient just to consider the case $n=0$ (why?). Hence we require that

$$\langle \varphi(t-k), \psi(t) \rangle = 0$$

for all k. Let us again first state a technical result.

Lemma 6.2. *Let φ be a scaling function, ψ the associated wavelet and define*

$$b(\omega) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(\omega + 2\pi n)\overline{\hat{\psi}(\omega + 2\pi n)}$$

Then $b(\omega) = 0$ for all ω.

Proof. Note again that b is 2π–periodic. Hence it has a Fourier series $b(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega}$ where

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_0^{2\pi} b(\omega)e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} \hat{\varphi}(\omega+2\pi n)\overline{\hat{\psi}(\omega+2\pi n)}e^{-ik\omega}d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} \hat{\varphi}(\omega+2\pi n)\overline{\hat{\psi}(\omega+2\pi n)}e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\omega)\overline{\hat{\psi}(\omega)}e^{-ik\omega}d\omega \end{aligned}$$

But by Parseval's theorem

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\omega)\overline{\hat{\psi}(\omega)}e^{-ik\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\omega)e^{-ik\omega}\overline{\hat{\psi}(\omega)}d\omega = \\ &\int_{-\infty}^{\infty}\varphi(t-k)\psi(t)dt = 0 \end{aligned}$$

where the final equality follows from the orthogonality of the subspaces V_0 and W_0 (Definitions 6.1 and 6.2). □

Theorem 6.4. *Let φ be a scaling function and ψ the associated wavelet. Then*

(i) $\langle h, S_{2n}g \rangle = 0$ for all n

(ii) $m_0(\omega)\overline{m_1(\omega)} + m_0(\omega+\pi)\overline{m_1(\omega+\pi)} = 0$

Proof.

$$\varphi(t) = 2\sum_{k=-\infty}^{\infty} h_k \varphi(2t-k)$$
$$\psi(t-n) = 2\sum_{k=-\infty}^{\infty} g_k \varphi(2t-2n-k) = 2\sum_{k=-\infty}^{\infty} g_{k-2n}\varphi(2t-k)$$

Hence

$$\langle \varphi(t), \psi(t-n)\rangle = 4\Big\langle \sum_{j=-\infty}^{\infty} h_j\varphi(2t-j), \sum_{k=-\infty}^{\infty} g_{k-2n}\varphi(2t-k)\Big\rangle =$$
$$4\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_j g_{k-2n}\langle \varphi(2t-j), \varphi(2t-k)\rangle = 2\sum_{k=-\infty}^{\infty} h_k g_{k-2n} = 2\langle h, S_{2n}g\rangle$$

The first result then follows from the orthogonality of subspaces V_0 and W_0. To prove the second statement we use Lemmas 6.1 and 6.2, and the fact that $\hat{\psi}(2\omega) = m_1(\omega)\hat{\varphi}(\omega)$.

$$0 = b(2\omega) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(2\omega+2\pi n)\overline{\hat{\psi}(2\omega+2\pi n)} = \sum_{n=-\infty}^{\infty} m_0(\omega+\pi n)\hat{\varphi}(\omega+\pi n)\overline{m_1(\omega+\pi n)\hat{\varphi}(\omega+\pi n)}$$
$$= \sum_{n=-\infty}^{\infty} m_0(\omega+2\pi n)\overline{m_1(\omega+2\pi n)}|\hat{\varphi}(\omega+2\pi n)|^2 +$$
$$\sum_{n=-\infty}^{\infty} m_0(\omega+\pi+2\pi n)\overline{m_1(\omega+\pi+2\pi n)}|\hat{\varphi}(\omega+\pi+2\pi n)|^2$$
$$= m_0(\omega)\overline{m_1(\omega)}\sum_{n=-\infty}^{\infty}|\hat{\varphi}(\omega+2\pi n)|^2 + m_0(\omega+\pi)\overline{m_1(\omega+\pi)}\sum_{n=-\infty}^{\infty}|\hat{\varphi}(\omega+\pi+2\pi n)|^2$$
$$= m_0(\omega)\overline{m_1(\omega)}a(\omega) + m_0(\omega+\pi)\overline{m_1(\omega+\pi)}a(\omega+\pi)$$
$$= m_0(\omega)\overline{m_1(\omega)} + m_0(\omega+\pi)\overline{m_1(\omega+\pi)}$$

□

We have seen that $\{S_{2n}h\}_{n\in\mathbb{Z}}$ and $\{S_{2n}g\}_{n\in\mathbb{Z}}$ are orthogonal sets in $l^2(\mathbb{Z})$. One can show that together they span the whole $l^2(\mathbb{Z})$. More precisely

Theorem 6.5. *Let φ be a scaling function and ψ the associated wavelet. Then*

$$\{S_{2n}h\}_{n\in\mathbb{Z}} \cup \{S_{2n}g\}_{n\in\mathbb{Z}}$$

is an orthognal basis in $l^2(\mathbb{Z})$.

It remains to find filters/transfer functions which satisfy the conditions which we have obtained. The first step is: given h and m_0, what are the corresponding g and m_1. Now it turns out that there is a canonical choice:

$$g_k = (-1)^k h_{1-k} \qquad\qquad m_1(\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)} \tag{6.7}$$

One can easily check that with this choice, the conditions of Theorem 6.4 are satisfied. In the following we will always suppose that h, g, m_0 and m_1 are related this way. Note in particular that this implies that

$$|m_0(\omega)|^2 + |m_1(\omega)|^2 = 1$$