Wavelets, spring 2002

7 Multiresolution analysis (continued)

So to find suitable wavelets we have to first find a good h or equivalently m_0 whose characterizing properties are:

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1$$

$$\langle h, S_{2n}h\rangle = \begin{cases} 1/2\ , & n = 0 \\ 0\ , & n \neq 0 \end{cases}$$

$$m_0(0) = \sum_{k=-\infty}^{\infty} h_k = 1$$

However, there are still two important practical points. Normally it is important that the wavelet and the scaling function have some smoothness, for example we may require that φ and ψ should be at least $k+\alpha$ –Lipschitz for some k and α. Also it is usually important that the wavelet has some vanishing moments. Although these two things are not directly related it turns out that requiring more vanishing moments tends to make the wavelet more regular as well. So let us first recall that having n vanishing moments means

$$\int_{-\infty}^{\infty} t^k \psi(t)dt = 0 \qquad 0 \leq k < n$$

In the Fourier domain this means that

$$\hat{\psi}^{(k)}(0) = 0 \qquad 0 \leq k < n$$

Now $\hat{\psi}(2\omega) = m_1(\omega)\hat{\varphi}(\omega)$ so that

$$2\hat{\psi}'(2\omega) = m_1'(\omega)\hat{\varphi}(\omega) + m_1(\omega)\hat{\varphi}'(\omega)$$

which implies that $2\hat{\psi}'(0) = m_1'(0)\hat{\varphi}(0) + m_1(0)\hat{\varphi}'(0) = m_1'(0)$. But because $m_1(\omega) = -e^{-i\omega}m_0(\omega+\pi)$ we get that $m_1'(0) = -m_0'(\pi)$. In other words we get

$$\hat{\psi}'(0) = 0 \quad \Leftrightarrow \quad m_0'(\pi) = 0$$

Continuing by induction we get

Theorem 7.1. *Suppose that*

1. $\psi \in C^{n-1}(\mathbb{R})$

2. *there exist $C > 0$ and $\varepsilon > 0$ such that*

$$|\psi(t)| \leq \frac{C}{(1+|t|)^{n+\varepsilon}}$$

3. $\psi^{(k)}$ *is bounded for* $0 \leq k < n$

Then ψ *has* n *vanishing moments if and only if*

$$m_0^{(k)}(\pi) = 0 \qquad 0 \leq k < n$$

In other words m_0 has a zero of multiplicity n at π. But then we can write

$$\begin{aligned} m_0(\omega) &= \Big(\frac{1+e^{-i\omega}}{2}\Big)^n \tilde{m}(\omega) \\ \tilde{m}(\omega) &= \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega} \end{aligned} \tag{7.1}$$

Clearly if m_0 is of this form, then it satisfies the condition in the above Theorem. Vanishing moments have another interesting consequence: one may represent polynomials exactly, up to certain degree. This is a bit surprising because scaling functions can be rather irregular, so one doesn't really expect that it is possible to get polynomials just by taking certain linear combinaisons.

Theorem 7.2. *Suppose that* ψ *has* n *vanishing moments and let* φ *be the corresponding scaling function. Then*

$$\sum_{k=-\infty}^{\infty} k^m \varphi(t-k)$$

is a polynomial of degree m *for* $0 \leq m < n$*.*

The exact determination of regularity of φ and ψ is very complicated. However, the following result gives a reasonable estimate.

Theorem 7.3. *Suppose that* m_0 *is given by* (7.1) *and let* $b = \max_\omega |\tilde{m}(\omega)|$*. Then* φ *and* $\psi \in C^{k+\alpha}(\mathbb{R})$ *if*

$$k + \alpha < n - \log_2 b - 1/2$$

Usually in practise b doesn't grow so fast so that increasing n indeed tends to increase the regularity of wavelets and scaling functions.

7.1 Sufficient conditions

Up to now we have supposed that we have φ which satisfies the conditions of MRA, and we have derived some consequences of this. However, if we reverse the process, namely:

- choose n, and let m_0 be as in (7.1)
- find the coefficients a_k such that $|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1$
- define $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega)$
- compute φ by taking the inverse Fourier transform of $\hat{\varphi}$

But does this in fact produce a MRA? In fact we must add two technical conditions. Up to now we have assumed that m_0 is continuous. In fact it must be a bit more regular so that the infinite product converges nicely. So we require that m_0 is α –Lipschitz for some $\alpha > 0$. In terms of h this is the same as requiring

$$\sum_{k=-\infty}^{\infty} |k|^{\alpha}|h_k| < \infty$$

In practice this is not restrictive at all. For example if we have a filter with only finite number of nonzero h_k, then m_0 is even infinitely differentiable, and the above sum has only finite number of terms. The other condition is more mysterious: we require that $|m_0(\omega)| \neq 0$ for $|\omega| \leq \pi/2$.

Theorem 7.4. *Suppose that*

(i) $m_0 \in C^{\alpha}(\mathbb{R})$ *for some* $\alpha > 0$

(ii) $|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$ *and* $m_0(0) = 1$

(iii) $|m_0(\omega)| \neq 0$ *for* $|\omega| \leq \pi/2$

Define $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega)$ *and let* φ *be the inverse Fourier transform of* $\hat{\varphi}$. *Then* φ *defines a MRA.*

7.2 Compact support

From practical point of view it would desirable to have φ and ψ with compact support, and filters h and g with only finite number of nonzero terms. In other words h and g should be FIR filters (finite impulse response). These two things are in fact equivalent. The other implication is rather easy to see.

Lemma 7.1. *Let* φ *have a compact support. Then* h *has only finite number of nonzero terms.*

Proof. Let $\mathrm{supp}(\varphi) \subset [-r, r]$. By property (2) of MRA we have

$$\tfrac{1}{2}\varphi(t/2) = \sum_{k=-\infty}^{\infty} h_k \varphi(t-k)$$

Taking inner product of both sides we obtain

$$\tfrac{1}{2}\langle \varphi(t/2), \varphi(t-n) \rangle = \langle \sum_{k=-\infty}^{\infty} h_k \varphi(t-k), \varphi(t-n) \rangle = \sum_{k=-\infty}^{\infty} h_k \langle \varphi(t-k), \varphi(t-n) \rangle = h_n$$

Now $\mathrm{supp}(\varphi(t-n)) \subset [-r+n, r+n]$ and $\mathrm{supp}(\varphi(t/2)) \subset [-2r, 2r]$. Hence $h_n = 0$ if $n < -3r$ or $n > 3r$ because in that case

$$\mathrm{supp}(\varphi(t-n)) \cap \mathrm{supp}(\varphi(t/2)) = \emptyset$$

□

Now it turns out that the converse is also true. This is not at all obvious, and the proof requires some rather advanced things and cannot be presented here. However, the result is important.

Theorem 7.5. *Let* $m_0(\omega) = \sum_{k=N_1}^{N_2} h_k e^{-i\omega k}$, $m_0(0) = 1$ *and define* $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(2^{-j}\omega)$. *Then*

- *supp*$(\varphi) \subset [N_1, N_2]$.
- *if* ψ *is the associated wavelet defined in the standard way, then the length of supp*(ψ) *is also* $N_2 - N_1$.