Wavelets, spring 2002

8 Signal analysis and reconstruction

8.1 Notation

Recall the projections

$$
\begin{aligned}
P_j \,:\, L^2(\mathbb{R}) \to V_j \qquad & P_j f = \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j,k}\rangle \varphi_{j,k} = \sum_{k=-\infty}^{\infty} c_{j,k}\varphi_{j,k} \\
Q_j \,:\, L^2(\mathbb{R}) \to W_j \qquad & Q_j f = \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k} = \sum_{k=-\infty}^{\infty} d_{j,k}\psi_{j,k}
\end{aligned}
\tag{8.1}
$$

Let us also define sequences

$$
\begin{aligned}
c^j =& \big(\dots, c_{j,-1}, c_{j,0}, c_{j,1}, c_{j,2}, \dots\big) \\
d^j =& \big(\dots, d_{j,-1}, d_{j,0}, d_{j,1}, d_{j,2}, \dots\big)
\end{aligned}
$$

Also for any sequence x we define $\tilde{x}$ by $\tilde{x}_n = x_{-n}$.

Definition 8.1. *If x is a sequence, the operators $\downarrow 2$,* downsampling*, and $\uparrow 2$,* upsampling*, are defined by*

$$
\begin{aligned}
y =& (\downarrow 2)x \quad \Leftrightarrow \quad y_n = x_{2n} \\
y =& (\uparrow 2)x \quad \Leftrightarrow \quad y_n = \begin{cases} x_k\ , & n = 2k \\ 0\ , & n = 2k+1 \end{cases}
\end{aligned}
$$

Note that $(\downarrow 2)(\uparrow 2)x = x$ for all x, but in general $(\uparrow 2)(\downarrow 2)x \neq x$. They are also *adjoints* (or transposes) of each other.

Definition 8.2. *Let H be a Hilbert space and $A : H \to H$ a linear operator. Operator B is said to be* adjoint *of A if for all x, $y \in H$*

$$\langle Ax, y\rangle = \langle x, By\rangle$$

The adjoint is of A is usually denoted by A^*.

Now upsampling and downsampling can be taken to be operators in $l^2(\mathbb{Z})$ and one can easily check that

$$\langle (\uparrow 2)x, y\rangle = \langle x, (\downarrow 2)y\rangle$$

8.2 Fast wavelet transform

Let us consider the decompositions

$$V_p = V_{p-1} \oplus W_{p-1} = \cdots = V_m \oplus W_m \oplus W_{m+1} \oplus \cdots \oplus W_{p-1} \tag{8.2}$$

Theorem 8.1. *Let h and g be the lowpass and highpass filters associated to MRA and let $c_{j,k}$ and $d_{j,k}$ be the coefficients of some signal f defined by the projections in* (8.1)*. Then*

$$c_{j,k} = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{n-2k} c_{j+1,n}$$
$$d_{j,k} = \sqrt{2} \sum_{n=-\infty}^{\infty} g_{n-2k} c_{j+1,n}$$

In vector notation we have

$$c^j = \sqrt{2}\ (\downarrow 2)\ \left(\tilde{h} * c^{j+1}\right)$$
$$d^j = \sqrt{2}\ (\downarrow 2)\ \left(\tilde{g} * c^{j+1}\right)$$

Proof. Since $\varphi_{j,k} \in V_j \subset V_{j+1}$ we can write

$$\varphi_{j,k}(t) = \sum_{n=-\infty}^{\infty} \langle \varphi_{j,k}, \varphi_{j+1,n} \rangle \varphi_{j+1,n}(t)$$

But

$$\langle \varphi_{j,k}, \varphi_{j+1,n} \rangle = \int_{-\infty}^{\infty} \varphi_{j,k}(t) \varphi_{j+1,n}(t) dt = \sqrt{2} 2^j \int_{-\infty}^{\infty} \varphi(2^j t - k) \varphi(2^{j+1} t - n) dt$$

By the change of variable $s = 2^{j+1}t - 2k$, $ds = 2^{j+1}dt$ we get

$$\langle \varphi_{j,k}, \varphi_{j+1,n} \rangle = \tfrac{1}{2} \int_{-\infty}^{\infty} \varphi(s/2) \varphi(s + 2k - n) ds = \sqrt{2} h_{n-2k}$$

So we obtain

$$\varphi_{j,k}(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{n-2k}\ \varphi_{j+1,n}(t)$$

Then taking the inner products on both sides gives

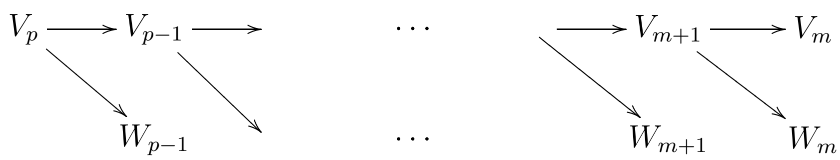
$$c_{j,k} = \langle f, \varphi_{j,k} \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{n-2k}\ \langle f, \varphi_{j+1,n} \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{n-2k} c_{j+1,n}$$

The proof of the formula for $d_{j,k}$ is entirely analogous. To get the vector formula we note that

$$c^j_k = c_{j,k} = \sum_{n=-\infty}^{\infty} h_{n-2k} c_{j+1,n} = \sum_{n=-\infty}^{\infty} \tilde{h}_{2k-n} c_{j+1,n} = \left(\tilde{h} * c^{j+1}\right)_{2k}$$

$\square$

With this Theorem we can compute recursively the wavelet coefficients, or in other words we can go from left to right in the equation (8.2). This yields the following diagram, which is sometimes called the *wavelet tree.*



This process is called the analysis of the signal because we hope that the computed coefficients give the desired information about the signal.

Theorem 8.2. *Let h and g be the lowpass and highpass filters associated to MRA and let $c_{j,k}$ and $d_{j,k}$ be the coefficients of some signal f defined by the projections in* (8.1)*. Then*

$$c_{j+1,k} = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{k-2n} c_{j,k} + \sqrt{2} \sum_{n=-\infty}^{\infty} g_{k-2n} d_{j,k}$$

In vector notation we have

$$c^{j+1} = \sqrt{2}\; h * \big((\uparrow 2) c^j\big) + \sqrt{2}\; g * \big((\uparrow 2) d^j\big)$$

Proof. Now $\varphi_{j+1,k} \in V_{j+1} = V_j \oplus W_j$ so we can write

$$\varphi_{j+1,k}(t) = \sum_{n=-\infty}^{\infty} \langle \varphi_{j+1,k}, \varphi_{j,n} \rangle \varphi_{j,n}(t) + \sum_{n=-\infty}^{\infty} \langle \varphi_{j+1,k}, \psi_{j,n} \rangle \psi_{j,n}(t)$$

But in the previous proof we saw that $\langle \varphi_{j+1,k}, \varphi_{j,n} \rangle = \sqrt{2} h_{k-2n}$ and $\langle \varphi_{j+1,k}, \psi_{j,n} \rangle = \sqrt{2} g_{k-2n}$. Hence

$$\begin{aligned} c_{j+1,k} =& \langle f, \varphi_{j+1,k} \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} h_{k-2n} \; \langle f, \varphi_{j,n} \rangle + \sqrt{2} \sum_{n=-\infty}^{\infty} g_{k-2n} \; \langle f, \psi_{j,n} \rangle \\ =& \sqrt{2} \sum_{n=-\infty}^{\infty} h_{k-2n} c_{j,n} + \sqrt{2} \sum_{n=-\infty}^{\infty} g_{k-2n} d_{j,n} \end{aligned}$$

To get the vector formula we use

$$\sum_{n=-\infty}^{\infty} h_{k-2n} c_{j,n} = \sum_{n=-\infty}^{\infty} h_{k-2n} \big((\uparrow 2) c^j\big)_{2n} = \sum_{n=-\infty}^{\infty} h_{k-n} \big((\uparrow 2) c^j\big)_n = \Big(h * \big((\uparrow 2) c^j\big)\Big)_k$$

□

This Theorem allows us to reverse the arrows in the wavelet tree, i.e. we can go also from right to left in the equation (8.2).

This process is called *synthesis* or *reconstruction* of the signal.
It remains to see how to start the process, i.e. when we have chosen m and p, how do we get c^p in the first place? It turns out that this is really relatively easy if we have sufficiently samples of the signal.
Let us start with simple

Lemma 8.1. *Let* $\alpha_j(t) = 2^j\varphi(2^j t)$ *and suppose that* $\text{supp}(\varphi) \subset [-r, r]$. *Then*

(i) $\int_{-\infty}^{\infty} \alpha_j(t)dt = 1$ *for all* j

(ii) $||\alpha_j||_1 = ||\varphi||_1$ *for all* j

(iii) $\text{supp}(\alpha_j) \subset [-2^{-j}r, 2^{-j}r]$

Proof. This is easy to verify. □

Now the sequence of functions α_j can be interpreted as Dirac's δ in the following sense.

Lemma 8.2. *Let* f *be continuous. Then*

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} f(t)\alpha_j(t)dt = f(0)$$

Proof.

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty} f(t)\alpha_j(t)dt - f(0)\Big| = \Big|\int_{-\infty}^{\infty} \big(f(t) - f(0)\big)\alpha_j(t)dt\Big| \leq \\
&\int_{-\infty}^{\infty} \big|f(t) - f(0)\big|\ |\alpha_j(t)|dt = \int_{-2^{-j}r}^{2^{-j}r} \big|f(t) - f(0)\big|\ |\alpha_j(t)|dt \leq \\
&\max_{|t|\leq 2^{-j}r} \big|f(t) - f(0)\big| \int_{-2^{-j}r}^{2^{-j}r} |\alpha_j(t)|dt = \max_{|t|\leq 2^{-j}r} \big|f(t) - f(0)\big|||\varphi||_1
\end{aligned}$$

Because f is continuous, the final expression tends to zero as j tends to infinity. □

This leads immediately to

Corollary 8.1. *Let* f *be continuous. Then*

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} f(t \pm s)\alpha_j(s)ds = f(t)$$

So let us be given a signal $f \in L^2(\mathbb{R})$. In practice we only have samples $f_k = f(k\Delta t)$ where Δt is the sampling period. So the problem is:

given samples f_k, how to compute or estimate c^p ?

So let us try to compute c^p.

$$\begin{aligned} c_k^p =& \langle f, \varphi_{p,k} \rangle = \int_{-\infty}^{\infty} f(t)\varphi_{p,k}(t)dt = 2^{p/2} \int_{-\infty}^{\infty} f(t)\varphi(2^p t - k)dt \\ =& 2^{p/2} \int_{-\infty}^{\infty} f(s + 2^{-p}k)\varphi(2^p s)ds = 2^{-p/2} \int_{-\infty}^{\infty} f(s + 2^{-p}k)\alpha_p(s)ds \end{aligned}$$

So putting $a = 2^{-p}k$ and using Corollary 8.1 we get

$$2^{p/2} c_{2^p a}^p = \int_{-\infty}^{\infty} f(s + a)\alpha_p(s)ds \to f(a)$$

when $p \to \infty$. In other words

$$f(2^{-p}k) \approx 2^{-p/2} c_k^p$$

when p is sufficiently "big". So in practice one scales the time appropriately and one chooses $\Delta t = 2^{-p}$ for some p. Then given the samples f_k one simply puts $c_k^p := 2^{p/2} f_k$.