Wavelets, spring 2002

9 Subband coding and filter banks

Let us recall the formulas

$$
\begin{aligned}
c^j &= \sqrt{2}\ (\downarrow 2)\ \left(\tilde{h} * c^{j+1}\right) \\
d^j &= \sqrt{2}\ (\downarrow 2)\ \left(\tilde{g} * c^{j+1}\right) \\
c^{j+1} &= \sqrt{2}\ h * \left((\uparrow 2) c^j\right) + \sqrt{2}\ g * \left((\uparrow 2) d^j\right)
\end{aligned}
$$

Let us for the moment forget that these represent the wavelet coefficients, and consider them as digital signals. So let us denote $x = c^{j+1}$, $u = c^j/\sqrt{2}$ and $v = d^j/\sqrt{2}$. With these notations we get

$$
\begin{aligned}
u &= \downarrow 2\ \left(\tilde{h} * x\right) \\
v &= \downarrow 2\ \left(\tilde{g} * x\right) \\
x &= 2\Big(\ h * \left(\uparrow 2\ u\right) + g * \left(\uparrow 2\ v\right)\Big)
\end{aligned}
$$

This may be summarised with the following diagram.

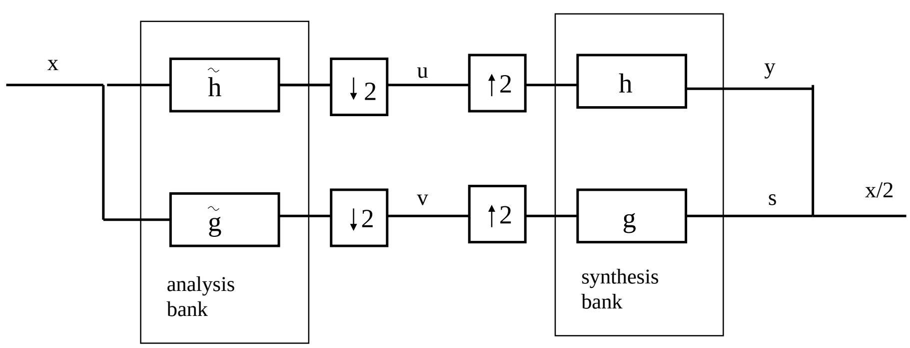

Figure 9.1: A filter bank with perfect reconstruction.

This is called subband coding because h is a low pass and g is a high pass filter, and therefore a given signal is separated into two frequency bands, or channels.
In the following it will be sometimes convenient to use $\mathcal{Z}$ – transform instead of Fourier–transform.

Definition 9.1. *The $\mathcal{Z}$ –transfrom of a sequence/signal x is*

$$X(z) = \sum_{k=-\infty}^{\infty} x_k z^{-k}$$

In particular we have $X(e^{i\omega}) = \hat{x}(\omega)$.

Lemma 9.1.

$$y = \downarrow 2\ x \quad \Leftrightarrow \quad \hat{y}(2\omega) = \tfrac{1}{2}\big(\hat{x}(\omega) + \hat{x}(\omega+\pi)\big) \quad \Leftrightarrow \quad Y(z^2) = \tfrac{1}{2}\big(X(z) + X(-z)\big)$$

Proof.

$$\hat{x}(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-i\omega k} = \sum_{k=-\infty}^{\infty} x_{2k} e^{-i2\omega k} + \sum_{k=-\infty}^{\infty} x_{2k+1} e^{-i\omega(2k+1)} = \hat{y}(2\omega) + e^{-i\omega} \sum_{k=-\infty}^{\infty} x_{2k+1} e^{-i2\omega k}$$

On the other hand

$$\hat{x}(\omega+\pi) = \hat{y}(2\omega) - e^{-i\omega} \sum_{k=-\infty}^{\infty} x_{2k+1} e^{-i2\omega k}$$

Adding these gives the first result. The formula about $\mathcal{Z}$ – transform is left as an exercise. □

So recalling that the transfer function of $\tilde{h}$ is $\tilde{m}_0(\omega) = \overline{m_0(\omega)}$ we can write

$$\begin{aligned}
\hat{u}(2\omega) &= \tfrac{1}{2}\left(\overline{m_0(\omega)}\hat{x}(\omega) + \overline{m_0(\omega+\pi)}\hat{x}(\omega+\pi)\right) \\
\hat{v}(2\omega) &= \tfrac{1}{2}\left(\overline{m_1(\omega)}\hat{x}(\omega) + \overline{m_1(\omega+\pi)}\hat{x}(\omega+\pi)\right) \\
U(z^2) &= \tfrac{1}{2}\left(M_0(z^{-1})X(z) + M_0(-z^{-1})X(-z)\right) \\
V(z^2) &= \tfrac{1}{2}\left(M_1(z^{-1})X(z) + M_1(-z^{-1})X(-z)\right)
\end{aligned}$$

Then we will need

Lemma 9.2.

$$y = \uparrow 2\ x \quad \Leftrightarrow \quad \hat{y}(\omega) = \hat{x}(2\omega) \quad \Leftrightarrow \quad Y(z) = X(z^2)$$

Proof.

$$\hat{y}(\omega) = \sum_{k=-\infty}^{\infty} y_k e^{-i\omega k} = \sum_{k=-\infty}^{\infty} x_k e^{-i2\omega k} = \hat{x}(2\omega)$$

□

This leads to formulas

$$\begin{aligned}
\hat{y}(\omega) &= m_0(\omega)\hat{u}(2\omega) & \qquad \hat{s}(\omega) &= m_1(\omega)\hat{v}(2\omega) \\
Y(z) &= M_0(z)U(z^2) & \qquad S(z) &= M_1(z)V(z^2)
\end{aligned}$$

Adding y and s (and multiplying by 2) gives then

$$\begin{aligned}
2\big(\hat{y}(\omega) + \hat{s}(\omega)\big) &= m_0(\omega)\big(\overline{m_0(\omega)}\hat{x}(\omega) + \overline{m_0(\omega+\pi)}\hat{x}(\omega+\pi)\big) + \\
&\quad m_1(\omega)\big(\overline{m_1(\omega)}\hat{x}(\omega) + \overline{m_1(\omega+\pi)}\hat{x}(\omega+\pi)\big) \\
&= \Big(|m_0(\omega)|^2 + |m_1(\omega)|^2\Big)\hat{x}(\omega) + \Big(m_0(\omega)\overline{m_0(\omega+\pi)} + m_1(\omega)\overline{m_1(\omega+\pi)}\Big)\hat{x}(\omega+\pi)
\end{aligned}$$

Hence to reproduce x we mush have

$$\begin{aligned} |m_0(\omega)|^2 + |m_1(\omega)|^2 &= 1 \\ m_0(\omega)\overline{m_0(\omega+\pi)} + m_1(\omega)\overline{m_1(\omega+\pi)} &= 0 \end{aligned} \tag{9.1}$$

The second equation is called *alias cancellation* and the first *perfect reconstruction*. Now recall that the standard choice for m_1 in wavelet theory is $m_1(\omega) = -e^{-i\omega}\overline{m_0(\omega+\pi)}$. With this choice clearly the second equation holds, and the first one becomes

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1 \tag{9.2}$$

which is the familiar condition for the construction of the scaling function.
Let's still write the relevant equations with help of $\mathcal{Z}$ – transfrom. The equations (9.1) yield

$$\begin{aligned} M_0(z)M_0(z^{-1}) + M_1(z)M_1(z^{-1}) &= 1 \\ M_0(z)M_0(-z^{-1}) + M_1(z)M_1(-z^{-1}) &= 0 \end{aligned} \tag{9.3}$$

The standard choice in $\mathcal{Z}$ – domain is $M_1(z) = -z^{-1}M_0(-z^{-1})$. Again it's easy to check that the second equation holds, and the first one gives

$$M_0(z)M_0(z^{-1}) + M_0(-z)M_0(-z^{-1}) = 1 \tag{9.4}$$

Let us put $P(z) = M_0(z)M_0(z^{-1})$. Then clearly $P(z) = P(z^{-1})$ which implies that $p_n = p_{-n}$ for all n. Moreover the condition (9.4) can be written as

$$P(z) + P(-z) = \sum_{k=-\infty}^{\infty} \big(p_k + (-1)^k p_k\big) z^{-k} = 1$$

So we must have $p_{2k} = 0$ for all $k \neq 0$, and $p_0 = 1/2$.

9.1 Generalization

Now we arrived at the situation in Figure 9.1 by MRA. But suppose that we are only interested in perfect reconstruction. So consider the situation in Figure 9.2.
By a straightforward computation we see that the equations (9.1) yield in this case

$$\begin{aligned} A(\omega)C(\omega) + B(\omega)D(\omega) &= 1 \\ A(\omega+\pi)C(\omega) + B(\omega+\pi)D(\omega) &= 0 \end{aligned} \tag{9.5}$$

and in $\mathcal{Z}$ – domain we get

$$\begin{aligned} A(z)C(z) + B(z)D(z) &= 1 \\ A(-z)C(z) + B(-z)D(z) &= 0 \end{aligned} \tag{9.6}$$

Obviously we have now much more freedom at our disposal. However, we still want only FIR filters. So which choices give FIR filters? Let us analyse the equations (9.6) more closely. We may view it as a system of equations for $C(z)$ and $D(z)$. So let us introduce the matrix

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ A(-z) & B(-z) \end{pmatrix}$$

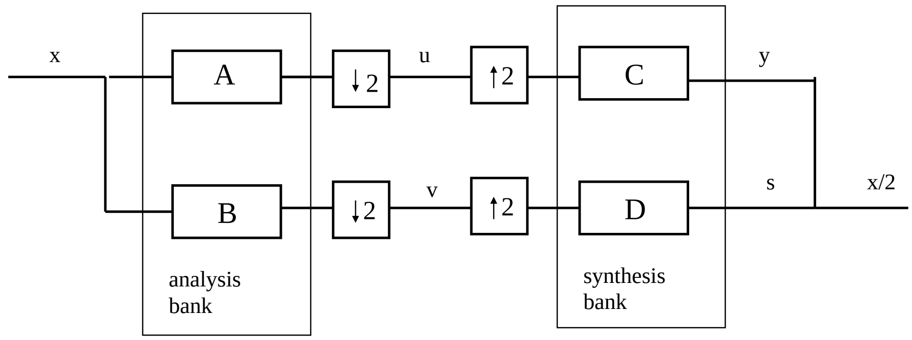

Figure 9.2: A more general filter bank with perfect reconstruction.

Lemma 9.3. *Suppose that the analysis bank, A and B are FIR filters. Then C and D will also be FIR if*

$$\det\big(M(z)\big) = A(z)B(-z) - A(-z)B(z) = \beta z^m$$

for some β and m.

Proof. The solution of (9.6) can be written as

$$\begin{pmatrix} C(z) \\ D(z) \end{pmatrix} = \big(M(z)\big)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\det\big(M(z)\big)}\begin{pmatrix} B(-z) & -B(z) \\ -A(-z) & A(z) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Using the condition on the determinant we get

$$\begin{aligned} C(z) &= z^{-m}B(-z)/\beta \\ D(z) &= -z^{-m}A(-z)/\beta \end{aligned} \tag{9.7}$$

Obviously these define FIR filters if A and B are FIR. □

In fact we have one obvious restriction:

Lemma 9.4. *The number m in Lemma 9.3 must be odd.*

Proof. Denote $Q(z) = A(z)B(-z) - A(-z)B(z)$. Then $Q(z) = -Q(-z)$, so that Q is an odd function. Hence m must be odd. □

Since the constant β is not important, let's just choose $\beta = 1$. So the design problem is: find A and B such that

$$A(z)B(-z) - A(-z)B(z) = z^m$$

It is convenient to reformulate this in another way. Using the first equation in (9.7) we obtain

$$A(z)C(z) + A(-z)C(-z) = 1$$

Let us define $P(z) = A(z)C(z)$. Then the above equation can be written as

$$P(z) + P(-z) = 1$$

Hence $p_0 = 1/2$ and $p_{2k} = 0$ if $k \neq 0$, but in this case in general $p_{2k+1} \neq p_{-2k-1}$.