Wavelets, spring 2002

10 Biorthogonal wavelets

Let us again consider the following filter bank:

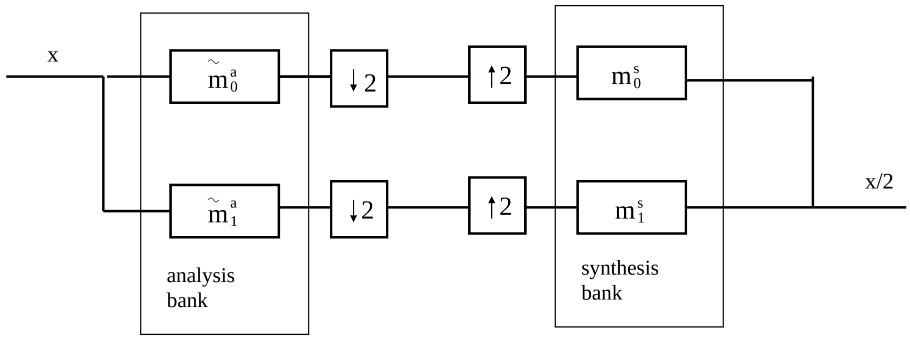

Figure 10.1: A general filter bank with perfect reconstruction.

To have the perfect reconstruction property with FIR filters the transfer functions must satisfy

$$\begin{aligned}
&\overline{m_0^a(\omega)}m_0^s(\omega) + \overline{m_0^a(\omega+\pi)}m_0^s(\omega+\pi) = 1 \\
&m_1^a(\omega) = -e^{-i\omega}\overline{m_0^s(\omega+\pi)} \\
&m_1^s(\omega) = -e^{-i\omega}\overline{m_0^a(\omega+\pi)}
\end{aligned}$$

We have taken $m = 1$ in Lemma 9.3. Now the ordinary MRA corresponds to the case $m_0 = m_0^a = m_0^s$, but here we have more possibilities. Anyway if we know all these filters then we can define

$$\begin{cases} \hat{\varphi}^a(\omega) = \prod_{j=1}^{\infty} m_0^a(2^{-j}\omega) \\ \hat{\psi}^a(2\omega) = m_1^a(\omega)\hat{\varphi}^a(\omega) \end{cases} \qquad \begin{cases} \hat{\varphi}^s(\omega) = \prod_{j=1}^{\infty} m_0^s(2^{-j}\omega) \\ \hat{\psi}^s(2\omega) = m_1^s(\omega)\hat{\varphi}^s(\omega) \end{cases} \tag{10.1}$$

We know that the infinite products converge, at least in the sense of distributions, and that taking then inverse Fourier transforms gives functions with compact support. But what properties do these functions have? Surely φ^a and φ^s do not define a MRA (except if $m_0^a = m_0^s$). Hence we must generalize the definition 6.1.

Definition 10.1. *MRA in $L^2(\mathbb{R})$ is a collection of subspaces $V_j \subset L^2(\mathbb{R})$ and a* scaling function *$\varphi \in L^2(\mathbb{R})$ such that*

(1) $\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots$

(2) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$

(3) $\cap_{j\in\mathbb{Z}} V_j = 0$

(4) $closure\big(\cup_{j\in\mathbb{Z}} V_j\big) = L^2(\mathbb{R})$

(5) $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ *is a Riesz basis of* V_0.

Let us further set $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k)$. One can then immediately check that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is a Riesz basis of V_j.
Now a Riesz basis is a kind of next best thing after orthonormal basis. But it's not very convenient to use. Consider $f \in V_0$; then we know that there must be constants c_k such that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi(t-k)$$

But how to compute these constants? To proceed we need

Definition 10.2. *Let* $B = \{e_k\}_{k\in\mathbb{Z}}$ *and* $\tilde{B} = \{\tilde{e}_k\}_{k\in\mathbb{Z}}$ *be bases of* $L^2(\mathbb{R})$. *Then* B *and* $\tilde{B}$ *are said to be* biorthogonal *if*

$$\langle e_k, \tilde{e}_m \rangle = \begin{cases} 1\ , & k = m \\ 0\ , & k \neq m \end{cases}$$

In this case we also say that bases B *and* $\tilde{B}$ *are* dual *to each other.*

Now if we have biorthogonal bases then we have two representations:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e_k(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \tilde{e}_k(t)$$

and by biorthogonality property the coefficients are easily calculated:

$$c_k = \langle f, \tilde{e}_k \rangle \qquad\qquad \tilde{c}_k = \langle f, e_k \rangle$$

So let's get back to our equations:

$$\begin{aligned} &\overline{m_0^a(\omega)} m_0^s(\omega) + \overline{m_0^a(\omega+\pi)} m_0^s(\omega+\pi) = 1 \\ &m_0^a(0) = m_0^s(0) = 1 \\ &\hat{\varphi}^a(\omega) = \prod_{j=1}^{\infty} m_0^a(2^{-j}\omega) \\ &\hat{\varphi}^s(\omega) = \prod_{j=1}^{\infty} m_0^s(2^{-j}\omega) \end{aligned} \tag{10.2}$$

So we could compute as easily as in the orthogonal case in the following case

Definition 10.3. *Let φ^a and φ^s be scaling functions of two MRAs. These MRAs are said to be biorthogonal, if*

$$\langle \varphi^a(t-k), \varphi^s(t-m) \rangle = \begin{cases} 1\,, & k = m \\ 0\,, & k \neq m \end{cases}$$

Now unfortunately all solutions of (10.2) do not yield biorthogonal MRAs. Let us state one sufficient condition.

Theorem 10.1. *Let m_0^a and m_0^s be solutions of* (10.2) *and consider the factored forms*

$$m_0^a(\omega) = \Big(\frac{1+e^{-i\omega}}{2}\Big)^{N_a} p_a(\omega)$$
$$m_0^s(\omega) = \Big(\frac{1+e^{-i\omega}}{2}\Big)^{N_s} p_s(\omega)$$

Define further:

$$B_k^a = \max_\omega \Big| \prod_{j=1}^{k} m_0^a(2^{-j}\omega) \Big|$$

$$B_k^s = \max_\omega \Big| \prod_{j=1}^{k} m_0^s(2^{-j}\omega) \Big|$$

Now if

$$\sup_k B_k^a < 2^{N_a - 1/2} \quad \textit{and} \quad \sup_k B_k^s < 2^{N_s - 1/2}$$

then φ^a and φ^s as defined by (10.2) *yield biorthogonal MRAs.*

We see again the importance of the number of vanishing moments.
Now supposing that the conditions of the Theorem are satisfied we can further define ψ^a and ψ^s as in (10.1). Then $\{\psi^a(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of W_0^a and $\{\psi^s(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of W_0^s. By scaling we can then define also W_j^a and W_j^s. Now V_j^a and W_j^a together span V_{j+1}^a like in orthogonal case, but they are *not* orthogonal to each other. On the other hand

$$V_j^a \perp W_j^s \qquad \text{and} \qquad V_j^s \perp W_j^a$$

So all in all we have the following properties:

$$\langle \varphi_{j,k}^a, \varphi_{j,m}^s \rangle = \begin{cases} 1\,, & k = m \\ 0\,, & k \neq m \end{cases}$$

$$\langle \psi_{j,k}^a, \psi_{l,m}^s \rangle = \begin{cases} 1\,, & k = m \text{ and } j = l \\ 0\,, & \text{otherwise} \end{cases}$$

Hence for a given signal f we have the following representations:

$$\begin{aligned} f(t) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi^a_{j,k} \rangle \psi^s_{j,k}(t) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi^s_{j,k} \rangle \psi^a_{j,k}(t) \\ &= \sum_{k=-\infty}^{\infty} \langle f, \varphi^a_{m,k} \rangle \varphi^s_{m,k}(t) + \sum_{j=m}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi^a_{j,k} \rangle \psi^s_{j,k}(t) \\ &= \sum_{k=-\infty}^{\infty} \langle f, \varphi^s_{m,k} \rangle \varphi^a_{m,k}(t) + \sum_{j=m}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi^s_{j,k} \rangle \psi^a_{j,k}(t) \end{aligned}$$

So in a biorthogonal case the coefficients in the representations are just as easily manipulated as in the orthogonal case.

11 Further reading

You can try some of the following books. The book by Daubechies [2] is a kind of classic. It explains the basic matters: frames, bases and the construction of compactly supported wavelets. Mallat's book [4] is less mathematical and covers more topics. Cohen was a graduate student of Meyer, and his book [1] is basically a translation of his thesis. Strang's book [8] treats more filter banks and subband coding than wavelet theory. It is quite readable. Jaffard and Meyer [3] discuss wavelets from various perspectives: historical, philosophical and practical. It is quite nicely written, and some parts of it are easily read while some passages are more technical. However, various chapters are mostly independent of each other. Finally there are the books by Meyer [5], [6], [7] which treat wavelets from functional analysis point of view. In other words wavelets are used to characterize various function spaces and operators in these spaces. These books have also been translated in English.

References

[1] A. Cohen and R. D. Ryan, *Wavelets and multiscale signal processing*, Chapman and Hall, 1995.

[2] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF conference series in applied mathematics, vol. 61, SIAM, 1992.

[3] S. Jaffard, Y. Meyer, and R. D. Ryan, *Wavelets, tools for science & technology*, revised ed., SIAM, 2001.

[4] S. Mallat, *A wavelet tour of signal processing*, Academic press, 1998.

[5] Yves Meyer, *Ondelettes et opérateurs. I*, Hermann, 1990.

[6] ———, *Ondelettes et opérateurs. II*, Hermann, 1990.

[7] Yves Meyer and R. R. Coifman, *Ondelettes et opérateurs. III*, Hermann, 1991.

[8] G. Strang and T. Nguyen, *Wavelets and filter banks*, Wellesley-Cambridge press, 1996.