Roots of complex equation $z^n = a$

Lehman - Pesonen 2007
http://www.joensuu.fi/mathematics/MathDistEdu/SemProd/CaenActivitiesNov2007.html

1. What are the solutions of $z^n = 1$ (for $n = 2, 3, 4$) ?

Let $z = x + iy = |z|e^{iC}$, with x, y and C real.
Play with the dynamic figure (called *sketch*) by pushing the buttons and moving z in the plane.

Problems 1

Reset by pushing the computer keyboard 'R' button.
1a) Give the two solutions of the equation $z^2 = 1$.
Answer 1a1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1a2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
1b) Give the four solutions of the equation $z^4 = 1$.
Answer 1b1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1b2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1b3): $z_3 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1b4): $z_4 = (\_\_\_\_) + i(\_\_\_\_)$
1c) The equation $z^3 = 1$ has three solutions: 1, j and j^2. Give approximative values of j and j^2:
Answer 1c1): $j = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1c2): $j^2 = (\_\_\_\_) + i(\_\_\_\_)$
1d) For which values of z do the points z and z^4 lay on top of each other?
Answer 1d1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1d2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1d3): $z_3 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1d4): $z_4 = (\_\_\_\_) + i(\_\_\_\_)$
1e) For which values of z do the points z^2 and z^4 lay on top of each other?
Answer 1e1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1e2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1e3): $z_3 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1e4): $z_4 = (\_\_\_\_) + i(\_\_\_\_)$
1f) Explain why there are less than 4 solutions.
Answer 1f): _____

2. What are the solutions of $z^n = a$ (for $n = 2, 3, 4$) ?

Let $z = x + iy = |z|e^{iC}$, with x, y and C real.
Play with the dynamic figure (called *sketch*) by pushing the buttons and moving z in the plane.

Problems 2

2a) Reset by pushing the computer keyboard 'R' button and do not move a. Use the variable point z to find the cartesian coordinates of a.
Answer 2a1): $\mathrm{Re}\, a = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2a2): $\mathrm{Im}\, a = (\_\_\_\_) + i(\_\_\_\_)$
2b) Use the variable point z to find the polar coordinates of a.
Answer 2b1): $\arg a = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2b2): $|a| = (\_\_\_\_) + i(\_\_\_\_)$
2c) Reset again. Click the buttons `Show z^2` and `|z| = |a|^1/2`.
Find the two solutions of $z^2 = a$.
Answer 2c1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2c2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
2d) Reset again. Click the button `Show z^3` and `|z| = |a|^1/3`.
Find the three solutions of $z^3 = a$.
Answer 2d1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2d2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2d3): $z_3 = (\_\_\_\_) + i(\_\_\_\_)$
2e) What can you say about the (algebraic) sum of the arguments of the solutions?
Answer 2e): _____
2f) Reset again. Click the button `Show z^4` and `|z| = |a|^1/4`.
Find the four solutions of $z^4 = a$.
Answer 2f1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2f2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2f3): $z_3 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2f4): $z_4 = (\_\_\_\_) + i(\_\_\_\_)$
2g) What can you say about z_2/z_1, z_3/z_1, z_4/z_1, z_2/z_4, z_1/z_3, z_p/z_q ?
Answer 2g): _____
2h) What is the shape of the quadrangle $z_1z_2z_3z_4$?
Answer 2h): _____

3. About the solutions of a complex number equation $z^x = a$, x real

Introduction to Problems 3

For a given complex number z there are q complex numbers that could be denoted $z^{p/q}$, namely the q solutions of the equation in w: $w^q = z^p$.
If x is an integer, there is only one complex number z^x, but if x is general real number, z^x may not mean anything.
Here the complex number denoted by z^x is the number $|z|^x e^{ixC}$, where C is the real number such that $\arg(z) = C + k2\pi$ and $-\pi < C < \pi$.
The locus shown on the screen is the curve $[0, x] \to \mathbf{C}$, $t \mapsto |z|^t e^{itC}$.

Problems 3

3a) Moving the point a, check that if $r = |z|$ then $|z^x| = r^x$.
3b) Choose $x = 0.5$; where are the values of a such that there is a complex number z for which z^x is on top of a ?
Answer 3b): _____
3c1) Choose $x = 0.75$; where are the points a for which two positions of z allows z^x is on top of a for some x ?
Answer 3c1): _____
3c2) Where are the points a for which there is only one position of z?
Answer 3c2): _____