About the roots of the equation $z^2 + bz + c = 0$

1. Where are the roots of the equation $z^2 + bz + c = 0$?

Problems 1

Reset by using the computer key 'R' (when the figure is clicked to be active).
To choose the equation coefficients b and c at random, move the four yellow points on the horizontal bottom lines.
1a) What is the value of b? Answer 1a): $b = (\_\_\_\_) + i(\_\_\_\_)$
1b) What is the value of c? Answer 1b): $c = (\_\_\_\_) + i(\_\_\_\_)$
1c) Click the button `Show z^2 + bz + c`.
Find the two solutions (roots) z_1 and z_2 of the equation

$$z^2 + bz + c = 0.$$

Hint. Move the free variable red point z on the plane area.
Answer 1c1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1c2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
1d) Compute $z_1 + z_2$ and $z_1 z_2$.
Answer 1d1): $z_1 + z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 1d2): $z_1 z_2 = (\_\_\_\_) + i(\_\_\_\_)$

2. About the roots and coefficients of $z^2 + bz + c = 0$

Problems 2

Reset by using the computer key 'R' (when the figure is clicked to be active).
Choose again b and c at random.
2a) What is the value of b? Answer 2a): $b = (\_\_\_\_) + i(\_\_\_\_)$
2b) What is the value of c? Answer 2b): $c = (\_\_\_\_) + i(\_\_\_\_)$
2c) Click the buttons `Show z^2` and `Show - bz - c`.
Find the two solutions z_1 and z_2 of the equation

$$z^2 + bz + c = 0.$$

Answer 2c1): $z_1 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2c2): $z_2 = (\_\_\_\_) + i(\_\_\_\_)$
2d) Compute $z_1 + z_2$ and $z_1 z_2$.
Answer 2d1): $z_1 + z_2 = (\_\_\_\_) + i(\_\_\_\_)$
Answer 2d2): $z_1 z_2 = (\_\_\_\_) + i(\_\_\_\_)$
How can you explain the relations between $z_1 + z_2$, $z_1 z_2$, b and c?

Answer: ____

3. The loci of $f(z) := z^2 + bz + c$ on segments

Introduction to tracing

In the dynamic figure beside you see a variable complex number z and its image $f(z)$ in the complex plane. Move z. When you click `Show z^2 + bz + c` and move z, you see traces that the image point $f(z)$ leaves behind itself. Tracing is stopped by the Hide button, but cleared from the screen by pushing the red cross on the lower right corner.

Problems 3

3a) Draw by tracing the image of the real number set.
Hint. Use tracing and move z along the horizontal axis, the real $x =$ Re z axis.
Finally, see the image of the x-axis under f by clicking the button `Image of x-axis`.
3b) Reset by using the computer key 'R', and choose again b and c at random (or leave them as they are).
Click the buttons `Show z1 and z2` and `Show segment z1z2`. You should see the segment connecting the *zeros* of f, i.e. the two numbers z_1 and z_2, for which $f(z_1) = 0 = f(z_2)$. Try with z. You also see a movable red point on the segment; it's there to help you.
Click the green buttons `Segment` and `Animate z on segment` (the animation is stopped by the same button).
When z describes the segment, the point $f(z) = z^2 + bz + c$ is describing its image, the curve called *locus*.
Click on the green `Locus` button to see the image $\{ f(z) \mid z \text{ on the segment} \}$.
Change the segment using its endpoints A and B. The locus is nearly always an arc of a parabola.
3c) What is the shape of the locus when it is not an arc of parabola?
Answer 3c): ______________________________
3d) When does it happen (use the red point on the segment)?
Answer 3d): ______________________________
3e) Click on `Show (z1+z2)/2`. Can you see whether you were right?
Answer 3e): ______________________________
3f) Hide z_1, z_2 and $\frac{1}{2}(z_1 + z_2)$. Can you imagine a new method of finding z_1 and z_2, using the segment and its image under f?
Answer 3f): ______________________________

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3g) Difficult question, not compulsory:
Prove that your answers above are always true.
Answer: ______________________________

4. The loci of $f(z) := z^2 + bz + c$ on circles

Problems 4

4a) Draw by tracing the image of the unit circle (UC).
See the locus of the unit circle under f by clicking the button `Image of UC`.
4b) Reset by using the computer key 'R', and choose b and c at random (or leave them as they are).
Click the buttons `Show segment z1z2` and `Show (z1+z2)/2`.
How can you verify that the numbers z_1 and z_2 really are the zeros of $f(z) := z^2 + bz + c$?
Answer 4b): ______________________________

4c). Click the gray buttons `Show circle` and `Animate z on circle`.
When z describes the circle, the point $f(z)$ describes its image, the curve called locus. Try tracing, stop it and clear the trace.
Click on the gray `Locus` button, and change the center and radius of the circle.
Are the loci of circles always circles?
Answer 4c): ______________________________

4d) What happens when you place the center of circle of animation on top of the midpoint of the segment connecting the zeros of f (the point $\frac{1}{2}(z_1 + z_2)$) ?
Answer 4d): ______________________________

4e) Place the circle of animation on top of the zeros of f. Can you imagine how to find the zeros just by using the loci of circles?
Answer 4e): ______________________________

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4f) The locus has a *double point* if there are two distinct points p_1 and p_2 so that $f(p_1) = f(p_2)$. Find out the circles whose locus has double points.
Answer 4f): ______________________________

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4g) Difficult question, not compulsory:
Prove that the locus of a circle under a polynomial $f(z) = z^2 + bz + c$ has a double point if and only if the midpoint $\frac{1}{2}(z_1 + z_2)$ of the zeros of f is inside the circle.
Answer 4g): ______________________________

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